Insensitivity of Quantized Hall Conductance to Disorder and Interactions

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A two-dimensional quantum Hall system is studied for a wide class of potentials including single-body random potentials and repulsive electron-electron interactions. We assume that there exists a non-zero excitation gap above the ground state(s), and then the conductance is derived from the linear perturbation theory with a sufficiently weak electric field. Under these two assumptions, we proved that the Hall conductance σ_{xy} and the diagonal conductance σ_{yy} satisfy $|\sigma_{xy} + e^2\nu/h| \le \text{const.}L^{-1/12}$ and $|\sigma_{yy}| \le \text{const.}L^{-1/12}$. Here e^2/h is the universal conductance with the charge $-e$ of electron and the Planck constant h; ν is the filling factor of the Landau level, and L is the linear dimension of the system. In the thermodymanic limit, our results show $\sigma_{xy} = -e^2\nu/h$ and $\sigma_{yy} = 0$. The former implies that integral and fractional filling factors ν with a gap lead to, respectively, integral and fractional quantizations of the Hall conductance.

KEY WORDS: Integral quantum Hall effect; fractional quantum Hall effect; Hall conductance; Landau Hamiltonian; random potential; electron-electron interaction.

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1 Introduction

The quantum Hall effect[1] is one of the most remarkable phenomena discovered in solid state physics. The effect is observed in two-dimensional electrons gases subjected to a uniform perpendicular magnetic field. Experimentally, such systems are realized at interfaces in semiconductors. The first experiments of the resistivity in a two-dimensional electron system in a magnetic field were performed by Kawaji, Igarashi and Wakabayashi [4] and Igarashi, Wakabayashi and Kawaji [5] in 1975. Unfortunately the quality of the samples in their early experiments[2] had not reached the stage where plateaus of finite width for the Hall resistivity could be obtained. In the same 1975, Ando, Matsumoto and Uemura [7] also were studying a two-dimensional electron system with disorder in a magnetic field, and theoretically predicted some aspects of the quantum Hall effect. However, they did not expect that the Hall resistivity of the plateaus is almost precisely quantized.

Soon after these studies, the integral quantum Hall effect was discovered [8, 9], and the fractional quantum Hall effect was subsequently discovered [10, 11]. Most aspects of the integral quantum Hall effect may be understood with an essentially single electron description, in which electron-electron interactions play only a secondary role. Actually early theoretical works [12, 14, 15] for explaining the integral quantum Hall effect were done along this line. In particular, a smashing idea of a topological invariant[3] for the conductance was introduced by Thouless, Kohmoto, Nightingale and den Nijs [14] and Kohmoto [15]. After their article [14], there appeared many variants [16, 17, 18, 19, 20] of their argument. In particular, some of the arguments were extended to a quantum Hall system with electron-electron interactions [17, 18, 19, 20]. However, the results always show an integral quantization of the Hall conductance without ad hoc assumptions [17]. It is questionable that the fractional quantization of the Hall conductance can be understood with a topological invariant of the Hall conductance.

For giving an explanation of the fractional quantum Hall effect, the difficulty comes from the fact that the electron-electron interaction is essential to this phenomenon. In order to overcome the difficulty, it is necessary to clarify the nature of the ground state(s) and of the low energy excitations for a strongly interacting electrons gas in a uniform magnetic field. For such a system, there appeared many approximate theories, trial functions for the ground state(s), perturbative approaches, mean field approximations, numerical analysis, etc [21]. However, the quantum Hall effect, in particular, the fractional quantization of the Hall conductance plateaus, is still not explained theoretically with a model of an interacting electrons gas in a uniform magnetic field.

In this paper, we consider a two-dimensional electrons gas in a uniform magnetic field with disorder and electron-electron interactions. The model is defined on an $L_x \times L_y$ rectangular box with periodic boundary conditions. The explicit form of the Hamiltonian is given by (2.1) in Section 2. We assume the existence of a non-zero excitation gap above the ground state(s). The existence of a gap is believed to be essential to the fractional

[1]For the history of the quantum Hall effect, see reviews [1, 2, 3].

[2]See also experiments [6].

[3]We should remark that Dubrovin and Novikov [13] preceded the article [14] and found a topologically nontrivial vector bundle and a topological invariant in a two-dimensional periodic Schrödinger operator with a magnetic field, although they did not treat the conductance.

quantization of the Hall conductance. Further we assume that an applied electric field to induce a Hall current is sufficiently weak so that the conductance is derived as the linear response coefficients from the linear perturbation theory. Under these two assumptions, we proved that the Hall conductance σ_{xy} and the diagonal conductance σ_{yy} satisfy

$$\left|\sigma_{xy} + \frac{e^2}{h}\nu\right| \le \text{const.} L_x^{5/24} L_y^{-7/24}, \quad |\sigma_{yy}| \le \text{const.} L_x^{5/24} L_y^{-7/24}. \tag{1.1}$$

Here ν is the filling factor of the Landau level, and e^2/h is the universal conductance with the charge $-e$ of electron and the Planck constant h. In particular,

$$\left|\sigma_{xy} + \frac{e^2}{h}\nu\right| \le \text{const.} L^{-1/12}, \quad |\sigma_{yy}| \le \text{const.} L^{-1/12} \quad \text{for } L_x = L_y = L. \tag{1.2}$$

Clearly we have

$$\sigma_{xy} = -\frac{e^2}{h}\nu, \quad \sigma_{yy} = 0 \tag{1.3}$$

in the thermodymanic limit $L \to +\infty$. In the next Section 2, the precise statements of these results and the precise definitions of the conductance σ_{xy}, σ_{yy} and of the filling factor ν will be given in a mathematically rigorous manner, and we will see that our results are justified for a wide class of potentials which includes single-body potentials with disorder and repulsive electron-electron interactions decaying by a power law. But the class does not include the standard Coulomb interaction proportional to $1/r$, where r is the distance between two electrons.

A reader may think that the finite-size corrections in the upper bounds of (1.2) are too large in comparison to the precision of the experimentally measured conductance. Actually the true finite-size corrections are expected to be exponentially small as in ref. [22]. But it is very hard to prove the corresponding statement in a mathematically rigorous manner.

Having the result (1.1) in mind, let us discuss which filling factor ν leads to a spectral gap above the ground state(s). For this purpose, we briefly state a result of our separate paper [23]. In the paper, we treated a two-dimensional quantum Hall system with electron-electron interactions and without disorder. The model is defined on an infinitely long strip with a large width, and the Hilbert space is restricted to the lowest $(n_{\max}+1)$ Landau levels with a large integer $n_{\max}$. The explicit form of the Hamiltonian is given in Section 2. In the infinite volume, we assumed the absence of non-translationally invariant infinite-volume ground state. Then we obtained the following result [23]:

- If a pure infinite-volume ground state has a non-zero excitation gap, then the filling factor ν must be equal to a rational number.[4]

Although we have considered the system without disorder, we can expect that, for the presence of weak disorder, the gap persists against the disorder. In this situation with the weak disorder, we get a rational quantization of the Hall conductance

$$\sigma_{xy} = -\frac{e^2}{h}\nu \quad \text{with a rational filling } \nu, \tag{1.4}$$

[4] In ref. [23], we also gave a phenomenological explanation for the reason why odd denominators of filling factions giving the quantized Hall conductance, are favor exclusively.

and the vanishing diagonal conductance

$$\sigma_{yy} = 0 \tag{1.5}$$

from the result (1.1). Further, in order to discuss the appearance of the Hall conductance plateaus, we change the filling factor ν slightly from the above rational value. Then, if the electrons of low energy excitations do not contribute to the current flow owing to the disorder, we can expect that the rationally quantized value of the Hall conductance σ_{xy} remains constant, i.e., there appears a plateau of the Hall conductance, with the vanishing diagonal conductance. The appearance of such a plateau will be discussed in relation to localization of wavefunctions in another separate paper [24].

This paper is organized as follows: In Section 2, we give the precise definition of the model and describe our main theorems in a mathematically rigorous manner. As preliminaries for the proofs of our theorems, we briefly review the eigenvalue problem of the single-electron Landau Hamiltonian and treat the Landau Hamiltonian with an electric field in Section 3. In Section 4, we discuss the relation between the electric potential of the present paper and the standard time-dependent vector potential. In Section 5, we calculate the current density by using the Rayleigh-Schrödinger perturbation theories with a sufficiently weak electric field, and give the proofs of our main theorems. For the convenience of readers, Appendices A-H are devoted to technical estimates and calculations of matrix elements appeared in our representation of the conductivities (conductance).

2 The model and the main results

We study a two-dimensional interacting N electrons system with a disorder potential V_ω in a unifrom magnetic field $(0,0,B)$ perpendicular to the x-y plane in which the electrons are confined, and in an electric field $(0,F,0)$ oriented along the y axis. For simplicity we assume that the electrons do not have spin degrees of freedom, although we can treat a quantum Hall system with spin degrees of freedom or with multiple layers in the same way. The Hamiltonian we consider in this paper is given by

$$\begin{aligned} H_\omega^{(N)} &= \sum_{j=1}^{N} \left[\frac{1}{2m_e} \left(p_{x,j} - eBy_j + A_0\right)^2 + \frac{1}{2m_e} p_{y,j}^2 + V_\omega(\mathbf{r}_j) \right] \\ &+ \sum_{1 \le i < j \le N} U^{(2)}(x_i - x_j, y_i - y_j) + \sum_{j=1}^{N} eFy_j P_{\text{bulk},j}, \end{aligned} \tag{2.1}$$

where $-e$ and m_e are, respectively, the charge of electron and the mass of electron, and A_0 is a real gauge parameter; $\mathbf{r}_j := (x_j, y_j)$ is the j th Cartesian coordinate of the N electrons. As usual, we define

$$p_{x,j} = -i\hbar \frac{\partial}{\partial x_j}, \quad \text{and} \quad p_{y,j} = -i\hbar \frac{\partial}{\partial y_j} \tag{2.2}$$

with the Planck constant $\hbar$. The system is defined on a rectangular box

$$S := [-L_x/2, L_x/2] \times [-L_y/2, L_y/2] \tag{2.3}$$

with periodic boundary conditions. We have intreoduced a projection operator $P_{\mathrm{bulk},j}$ so that the electrons near boundaries $y = \pm L_y/2$ do not feel the infinitely strong electric fields at the boundaries. Since the projection $(1 - P_{\mathrm{bulk},j})$ acts on a wavefunction at only a neighborhood of the boundaries, we can expect that the effect of the projection is negligible[5] in the thermodynamic limit $L_y \to +\infty$. In Section 3.2, we will give the precise definition of $P_{\mathrm{bulk},j}$, and show that the corresponding electric field is constant except for the neighborhood of the boundaries. In Section 4, we will discuss the relation between the regularized electric potential and the standard time-dependent vector potential. The latter yields the constant electric field on the whole torus.

We assume that the single-body potential V_ω with disorder satisfies the following conditions: periodic boundary conditions

$$V_\omega(x + L_x, y) = V_\omega(x, y + L_y) = V_\omega(x, y), \tag{2.4}$$

and

$$\|V_\omega\| < V_0 < \infty, \tag{2.5}$$

where V_0 is a positive constant which is independent of the linear dimensions L_x, L_y of the system. The potential V_ω consists of a random part V_ω^{ran} and a regular part W as

$$V_\omega(x, y) = V_\omega^{\mathrm{ran}}(x, y) + W(x). \tag{2.6}$$

The regular part W is a function of x only such that W satisfies

$$W(x + L_x) = W(x) = W(-x). \tag{2.7}$$

A simple example of W is given by[6]

$$W(x) = W_0 \cos \kappa x \quad \text{with } \kappa = \frac{2\pi}{L_x} n, \ n \in \mathbf{Z}, \tag{2.8}$$

where W_0 is a real constant.

The electron-electron interaction $U^{(2)}$ satisfies

$$U^{(2)}(-x, -y) = U^{(2)}(x, y). \tag{2.9}$$

We impose periodic boundary conditions as

$$U^{(2)}(x + L_x, y) = U^{(2)}(x, y + L_y) = U^{(2)}(x, y). \tag{2.10}$$

We assume that $U^{(2)}$ is two times continuously differentiable on $\mathbf{R}^2$, and satisfies

$$\left| \frac{\partial^2}{\partial x^2} U^{(2)}(x, y) \right| + \left| \frac{\partial^2}{\partial y^2} U^{(2)}(x, y) \right| \le \alpha U^{(2)}(x, y) \quad \text{for any } (x, y) \in \mathbf{R}^2, \tag{2.11}$$

[5] It is very difficult to give a proof for the statement that the boundary effect is negligible. In fact, we still cannot prove the claim.

[6] The question of the applicability of our method to a quantum Hall system with a periodic potential was brought to the author by Mahito Kohmoto. Thus we have partially answered his question, althuough we still cannot treat a periodic potential modulating in both x and y directions.

with a positive constant α which is independent of the linear dimensions L_x, L_y of the system. Further we assume that

$$U^{(2)}(x,y) \le U_0 \left\{1 + [\mathrm{dist}(x,y)/r_0]^2\right\}^{-\gamma/2} \quad \text{with } U_0 > 0,\ \gamma > 2,\ r_0 > 0, \tag{2.12}$$

where the distance is given by

$$\mathrm{dist}(x,y) := \sqrt{\min_{m\in\mathbf{Z}}\{|x - mL_x|^2\} + \min_{n\in\mathbf{Z}}\{|y - nL_y|^2\}}. \tag{2.13}$$

A simple example of $U^{(2)}$ satisfying these conditions is

$$U^{(2)}(x,y) = \frac{U_0}{[1 + (r/r_0)^2]^{\gamma/2}} \quad \text{with } \gamma > 2, \tag{2.14}$$

where U_0 and r_0 are posiitve constants, and

$$r = \sqrt{\left(\frac{L_x}{\pi}\right)^2 \sin^2\frac{\pi}{L_x}x + \left(\frac{L_y}{\pi}\right)^2 \sin^2\frac{\pi}{L_y}y}. \tag{2.15}$$

In the limit $L_x, L_y \to \infty$, we have the usual Euclidean distance $r = \sqrt{x^2 + y^2}$.

We take $L_xL_y = 2\pi M\ell_B^2$ with a sufficiently large positive integer M. Here ℓ_B is the so-called magnetic length defined as $\ell_B := \sqrt{\hbar/eB}$. The number M is equal to the number of the states in a single Landau level of the single-electron Hamiltonian in the uniform magnetic field with no single-body potential, and with no electric field. For simplicity, we take M even. We define the filling factor ν as $\nu = N/M$. We assume $\nu < \nu_0$, where ν_0 is a positive constant which is independent of L_x, L_y, N. The condition $L_xL_y = 2\pi M\ell_B^2$ for L_x, L_y is convenient for imposing the following periodic boundary conditions: For an N electrons wavefunction $\Phi^{(N)}$, we impose periodic boundary conditions

$$t_j^{(x)}(L_x)\Phi^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \Phi^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) \quad \text{for } j = 1, 2, \ldots, N, \tag{2.16}$$

and

$$t_j^{(y)}(L_y)\Phi^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \Phi^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) \quad \text{for } j = 1, 2, \ldots, N. \tag{2.17}$$

Here $t^{(x)}(\cdots)$ and $t^{(y)}(\cdots)$ are magnetic translation operators [25] defined as

$$t^{(x)}(x')f(x,y) = f(x - x', y), \quad t^{(y)}(y')f(x,y) = \exp[iy'x/\ell_B^2]f(x, y - y') \tag{2.18}$$

for a function f on $\mathbf{R}^2$, and a subscript j of an operator indicates that the operator acts on the j-th coordinate of a function.[7] The range of x' and y' are given by[8]

$$x' = m\Delta x \quad \text{with } m \in \mathbf{Z}, \quad \text{and} \quad y' = n\Delta y \quad \text{with } n \in \mathbf{Z}, \tag{2.19}$$

[7]Throughout the present paper, we use this convention.
[8]See Section 3.1.

where the minimal units of the translations are given by

$$\Delta x := \frac{h}{eB}\frac{1}{L_y}, \quad \text{and} \quad \Delta y := \frac{h}{eB}\frac{1}{L_x}. \tag{2.20}$$

For a given random potential V_ω, we define a set of random potentials as

$$\Omega(\omega) := \Omega^{\rm T}(\omega) \cup \Omega^{\rm R}(\omega) \tag{2.21}$$

with

$$\Omega^{\rm T}(\omega) := \left\{ \omega' \middle| V_{\omega'}(x,y) = V_\omega(x, y-y_0), y_0 = \frac{2\pi\hbar n}{eBL_x}, n \in \mathbf{Z} \right\} \tag{2.22}$$

and

$$\Omega^{\rm R}(\omega) := \left\{ \omega' \middle| V_{\omega'}(x,y) = V_\omega(-x, y_0-y), y_0 = \frac{2\pi\hbar n}{eBL_x}, n \in \mathbf{Z} \right\}. \tag{2.23}$$

Further we define an average with respect to the random potentials $\Omega(\omega)$ as

$$\mathbf{E}_\omega\left[\cdots\right] := \frac{1}{|\Omega(\omega)|}\sum_{\omega' \in \Omega(\omega)} (\cdots). \tag{2.24}$$

The regular part W of the single-body potential V_ω of (2.6) is invariant under the transformations in (2.22) and (2.23).

We denote by $H_{\omega,0}^{(N)}$ the Hamiltonian $H_\omega^{(N)}$ of (2.1) with $A_0 = 0$ and $F = 0$. We assume that the "ground state" of $H_{\omega,0}^{(N)}$ is finitely q-fold degenerate in the sense that the lowest-lying q energy eigenvalues $E_{\omega,(0,\mu)}^{(N)}$, $\mu = 1, 2, \ldots, q$ satisfy the condition

$$\Delta\mathcal{E} := \max_{\mu,\mu' \in \{1,2,\ldots,q\}} \left\{ \left| E_{\omega,(0,\mu)}^{(N)} - E_{\omega,(0,\mu')}^{(N)} \right| \right\} \to 0 \quad \text{as} \begin{cases} L_x, L_y \to \infty & ; \\ L_y \to \infty & \text{for a fixed } L_x, \end{cases} \tag{2.25}$$

where the limit is taken for a fixed filling $\nu = N/M$. Further we assume that there exists a non-zero excitation gap above the "ground state", i.e., the first excited state has an energy eigenvalue $E_{\omega,1}^{(N)}$ such that

$$\min_{\mu \in \{1,2,\ldots,q\}} \left\{ E_{\omega,1}^{(N)} - E_{\omega,(0,\mu)}^{(N)} \right\} \geq \Delta E, \tag{2.26}$$

where ΔE is a positive constant which is independent of L_x, L_y, N.

We denote by $\tilde{\Phi}_{\omega,(0,\mu)}^{(N)}$ with $\mu = 1, 2, \ldots, q$, the "ground state" eigenvectors of the Hamiltonian[9] $H_\omega^{(N)}$ of (2.1) with a sufficienlty weak electric field F. We take $\left\{\tilde{\Phi}_{\omega,(0,\mu)}^{(N)}\right\}$ to be an orthonormal system. Then the current density $\mathbf{j}$ at zero temperature is given by

$$j_s := -\frac{e^2}{L_x L_y}\frac{1}{q}\sum_{\mu=1}^{q} \mathbf{E}_\omega\left[\left\langle \tilde{\Phi}_{\omega,(0,\mu)}^{(N)}, v_{{\rm tot},s}\tilde{\Phi}_{\omega,(0,\mu)}^{(N)} \right\rangle\right] \quad \text{for } s = x, y, \tag{2.27}$$

[9] More precisely, we choose $A_0 = m_e F/B$ in addition to the condition of the small F. See Section 5.

where $\langle \cdots, \cdots \rangle$ stands for the inner product in the N electrons Hilbert space, and the velocity operator $\mathbf{v}_{\mathrm{tot}}$ for the N electrons is given by

$$v_{\mathrm{tot},s} := \begin{cases} \dfrac{1}{m_e} \displaystyle\sum_{j=1}^{N} (p_{x,j} - eBy_j + A_0) & \text{for } s = x; \\ \dfrac{1}{m_e} \displaystyle\sum_{j=1}^{N} p_{y,j} & \text{for } s = y. \end{cases} \tag{2.28}$$

The formula (2.27) for the current density $\mathbf{j}$ is justified for an inverse temperature β satisfying $\Delta\mathcal{E} \ll \beta^{-1} \ll \Delta E$. The conductivities are defined as

$$\sigma_{sy} := \lim_{F \downarrow 0} \frac{j_s}{F} \quad \text{for } s = x, y. \tag{2.29}$$

Now we describe our main theorems for both non-interacting and interacting electrons gases.

Theorem 2.1 *Suppose that there is no electron-electron interaction, i.e.,* $U^{(2)} = 0$, *and that there is a non-zero excitation gap above the "ground state" in the sense of (2.26). Then*

$$\left| \sigma_{xy} + \frac{e^2}{h}\nu \right| \le \mathcal{C}_{\mathrm{con},0} \left(\frac{\ell_B}{L_y} \right)^{3/5}, \quad |\sigma_{yy}| \le \mathcal{C}_{\mathrm{con},0} \left(\frac{\ell_B}{L_y} \right)^{3/5}, \tag{2.30}$$

where $\mathcal{C}_{\mathrm{con},0}$ *is a positive constant which is independent of the linear dimensions* L_x, L_y *of the system and of the number* N *of the electrons.*

We remark that the above assumption on the excitation gap is valid in the case with $\|V_\omega\| < V_0 < \hbar\omega_c/2$ and $\nu \in \{1, 2, \ldots\}$. Here ω_c is the cyclotron frequency given by $\omega_c := eB/m_e$. In fact the ground state is unique and has a non-zero excitation gap above it. In the thermodynamic limt $L_y \to \infty$, we have the integral quantization of the Hall conductance $\sigma_{xy} = -e^2(n+1)/h$ with the Landau level index $n = 0, 1, 2, \ldots$, and the vanishing diagonal conductance $\sigma_{yy} = 0$.

For the interacting electrons gas, we obtain the following theorem:

Theorem 2.2 *Suppose that the single-body potential* V_ω *is two times continuously differentiable on* $\mathbf{R}^2$ *and satisfies*

$$\left\| \frac{\partial^2}{\partial x^2} V_\omega \right\| + \left\| \frac{\partial^2}{\partial y^2} V_\omega \right\| < V_0' < \infty \tag{2.31}$$

with a positive constant V_0' *which is independent of the linear dimensions* L_x, L_y *of the system, and suppose that there is a non-zero excitation gap above the "ground state" in the sense of (2.26). Then there exists a positive number* $N_{\min}$ *such that* $N_{\min}$ *is independent of the linear dimensions* L_x, L_y *of the system and of the number* N *of the electrons, and that the following two bounds are valid for* $N \ge N_{\min}$*:*

$$\left| \sigma_{xy} + \frac{e^2}{h}\nu \right| \le \mathcal{C}_{\mathrm{con}} \left(\frac{L_x}{\ell_B} \right)^{5/24} \left(\frac{\ell_B}{L_y} \right)^{7/24}, \quad |\sigma_{yy}| \le \mathcal{C}_{\mathrm{con}} \left(\frac{L_x}{\ell_B} \right)^{5/24} \left(\frac{\ell_B}{L_y} \right)^{7/24}, \tag{2.32}$$

where $\mathcal{C}_{\rm con}$ *is a positive constant which is independent of the linear dimensions* L_x, L_y *of the system and of the number* N *of the electrons. In particular,*

$$\left|\sigma_{xy} + \frac{e^2}{h}\nu\right| \le \mathcal{C}_{\rm con}\left(\frac{\ell_B}{L}\right)^{1/12}, \quad |\sigma_{yy}| \le \mathcal{C}_{\rm con}\left(\frac{\ell_B}{L}\right)^{1/12} \quad \textit{for } L_x = L_y = L. \tag{2.33}$$

The number $N_{\rm min}$ is explicitly given as a function of the parameters of the model in (F.16) in Appendix F.2.

Having this result in mind, let us discuss which filling factor ν leads to a spectral gap above the "ground state". For this purpose, we briefly state a result of our separate paper [23]. Consider first the Hamiltonian

$$H^{(N)} := \sum_{j=1}^{N}\left\{\frac{1}{2m_e}\left[(p_{x,j} - eBy_j)^2 + p_{y,j}^2\right] + W(x_j)\right\} + \sum_{1\le i<j\le N} U^{(2)}(x_i - x_j, y_i - y_j) \tag{2.34}$$

which is the Hamiltonian (2.1) without the random potential $V_\omega^{\rm ran}$ and with $A_0 = 0$ and $F = 0$. Then we restrict the Hilbert space to the lowest $(n_{\rm max}+1)$ Landau level with a large integer $n_{\rm max}$. Namely the Hamiltonian we treated in ref. [23] is given by

$$H^{(N)}(n_{\rm max}) := P^{(N)}(n_{\rm max})H^{(N)}P^{(N)}(n_{\rm max}) \tag{2.35}$$

with the projection operator $P^{(N)}(n_{\rm max})$. We take the thermodynamic limit $L_y \to +\infty$ for a fixed large L_x and a fixed filling factor ν. In this infinite-volume limit, we assume the absence of non-translationally invariant infinite-volume ground state. Then we obtained the following result [23]:

- If a pure infinite-volume ground state has a non-zero excitation gap, then the filling factor ν must be equal to a rational number.[10]

Although the system we treated has no disorder, we can expect that, for the presence of weak disorder, the gap persists against the disorder. In this situation with the weak disorder, we get a rational quantization of the Hall conductance

$$\sigma_{xy} = -\frac{e^2}{h}\nu \quad \text{with a rational filling } \nu, \tag{2.36}$$

and the vanishing diagonal conductance

$$\sigma_{yy} = 0 \tag{2.37}$$

from the result (2.32).

In order to discuss the appearance of plateaus, we change the filling factor ν slightly from a rational value in the non-interacting or the interacting cases. Then, if the electrons of low energy excitations do not contribute to the current flow owing to the disorder, we can expect that the quantized value of the Hall conductance σ_{xy} remains constant, i.e., there appears a plateau of the Hall conductance, with the vanishing diagonal conductance. The appearance of such a plateau due to disorder will be discussed in relation to localization of wavefunctions in another separate paper [24].

[10] For the so-called odd denominator rule, see the footnote 4 in Section 1.

3 Single electron Landau systems

As preliminaries, we briefly review the properties of the single electron systems in a uniform magnetic field with no electric field, and then introduce an electric field.

3.1 The single electron Landau Hamiltonian

In this subsection, we briefly review the eigenvalue problem of the Landau Hamiltonian for a single electron in a uniform magnetic field. The Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2m_e}\left[(p_x - eBy)^2 + p_y^2\right]. \tag{3.1}$$

Consider first the eigenvalue problem on the infinite plane $\mathbf{R}^2$. In order to obtain an eigenvector of the Hamiltonian $\mathcal{H}$, put its form as

$$\phi(x,y) = e^{ikx}v(y) \tag{3.2}$$

with a wavenumber $k \in \mathbf{R}$. Substituting this into the Schrödinger equation $\mathcal{H}\phi = \mathcal{E}\phi$, one has

$$\left[\frac{1}{2m_e}(\hbar k - eBy)^2 + \frac{1}{2m_e}p_y^2\right]v(y) = \mathcal{E}v(y). \tag{3.3}$$

Clearly this is identical to the eigenvalue equation of a quantum harmonic oscillator as

$$\left[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial y^2} + \frac{e^2B^2}{2m_e}\left(y - \frac{\hbar k}{eB}\right)^2\right]v(y) = \mathcal{E}v(y). \tag{3.4}$$

The eigenvectors are

$$v_{n,k}(y) := v_n(y - y_k) := N_n \exp\left[-(y-y_k)^2/(2\ell_B^2)\right] H_n\left[(y-y_k)/\ell_B\right], \tag{3.5}$$

where H_n is the Hermite polynomial, $y_k = \hbar k/eB$, and N_n is the positive normalization constant so that

$$\int_{-\infty}^{+\infty} dy|v_{n,k}(y)|^2 = 1. \tag{3.6}$$

The energy eigenvalues are given by

$$\mathcal{E}_{n,k} := \left(n + \frac{1}{2}\right)\hbar\omega_c \quad \text{for} \quad n = 0,1,2,\ldots \tag{3.7}$$

with $\omega_c = eB/m_e$. Thus the eigenvectors of the Hamiltonian (3.1) are given by

$$\phi_{n,k}(x,y) = e^{ikx}v_{n,k}(y). \tag{3.8}$$

Next we consider single electron in $L_x \times L_y$ rectangular box $S = [-L_x/2, L_x/2] \times [-L_y/2, L_y/2]$ satisfying $L_xL_y = 2\pi M\ell_B^2$ with a sufficienlty large positive integer M. For simplicity we take M even. We impose periodic boundary conditions

$$\phi(x,y) = t^{(x)}(L_x)\phi(x,y), \quad \phi(x,y) = t^{(y)}(L_y)\phi(x,y) \tag{3.9}$$

for wavefunctions ϕ on $\mathbf{R}^2$. Here $t^{(x)}(\cdots)$ and $t^{(y)}(\cdots)$ are the magnetic translation operators defined by (2.18). We claim that the functions

$$f_1(x,y) = t^{(x)}(x')f(x,y) \tag{3.10}$$

and

$$f_2(x,y) = t^{(y)}(y')f(x,y) \tag{3.11}$$

satisfy the periodic boundary conditions (3.9) if f satisfies (3.9). As a result, x' and y' are restricted into the following values:

$$x' = m\Delta x \quad \text{with } m \in \mathbf{Z}, \quad \text{and} \quad y' = n\Delta y \quad \text{with } n \in \mathbf{Z}, \tag{3.12}$$

where

$$\Delta x := \frac{h}{eB}\frac{1}{L_y}, \quad \text{and} \quad \Delta y := \frac{h}{eB}\frac{1}{L_x}. \tag{3.13}$$

In fact one has

$$\begin{aligned} f_1(x,y) &= f(x-x',y) \\ &= \exp[iL_y(x-x')/\ell_B^2]f(x-x',y-L_y) \\ &= \exp[-iL_y x'/\ell_B^2]\exp[iL_y x/\ell_B^2]f(x-x',y-L_y) \\ &= \exp[-iL_y x'/\ell_B^2]\exp[iL_y x/\ell_B^2]f_1(x,y-L_y) \\ &= \exp[-iL_y x'/\ell_B^2]t^{(y)}(L_y)f_1(x,y) \\ &= \exp[-iL_y x'/\ell_B^2]f_1(x,y). \end{aligned} \tag{3.14}$$

by the definitions. This implies $L_y x'/\ell_B^2 = 2\pi m$ with an integer m. Similarly

$$\begin{aligned} f_2(x,y) &= \exp[iy'x/\ell_B^2]f(x,y-y') \\ &= \exp[iy'x/\ell_B^2]f(x-L_x,y-y') \\ &= \exp[iy'L_x/\ell_B^2]\exp[iy'(x-L_x)/\ell_B^2]f(x-L_x,y-y') \\ &= \exp[iy'L_x/\ell_B^2]f_2(x-L_x,y) \\ &= \exp[iy'L_x/\ell_B^2]t^{(x)}(L_x)f_2(x,y) \\ &= \exp[iy'L_x/\ell_B^2]f_2(x,y). \end{aligned} \tag{3.15}$$

Thus $y'L_x/\ell_B^2 = 2\pi n$ with an integer n. In the following we restrict the ranges of the variables x', y' in the magnetic translations to these values.

Since

$$t^{(y)}(y')(p_x - eBy)\left[t^{(y)}(y')\right]^{-1} = p_x - eBy \tag{3.16}$$

for any y', the Hamiltonian (3.1) is invariant under all the magnetic translations $t^{(x)}(\cdots)$ and $t^{(y)}(\cdots)$. Consider wavefunctions

$$\phi_{n,k}^{\mathrm{P}}(x,y) = L_x^{-1/2}\sum_{\ell=-\infty}^{+\infty} e^{i(k+\ell K)x}v_{n,k}(y-\ell L_y) \tag{3.17}$$

for $k = 2\pi m/L_x$ with $m = -M/2+1, \ldots, M/2-1, M/2$, and with $K = L_y/\ell_B^2$. These wavefunctions are the eigenvectors of the Hamiltonian (3.1) satisfying the periodic boundary conditions (3.9), because $L_x L_y = 2\pi M \ell_B^2$ with the integer M. The eigenvalues of $\phi_{n,k}^{\mathrm{P}}$ are given by (3.7).

We define a reflection operator R as

$$Rf(x,y) = f(-x,-y) \tag{3.18}$$

for a function on $\mathbf{R}^2$. One can easily get the following lemma:

Lemma 3.1 *The vector $\phi_{n,k}^{\mathrm{P}}$ of (3.17) is an eigenvector of the magnetic translation $t^{(x)}(\Delta x)$, i.e.,*

$$t^{(x)}(\Delta x)\phi_{n,k}^{\mathrm{P}} = e^{-ik\Delta x}\phi_{n,k}^{\mathrm{P}} = e^{-i2\pi m/M}\phi_{n,k}^{\mathrm{P}} \quad \textit{with} \quad k = \frac{2\pi m}{L_x}, \tag{3.19}$$

and the magnetic translation $t^{(y)}(\Delta y)$ shifts the wavenumber k of the vector $\phi_{n,k}^{\mathrm{P}}$ by one unit $2\pi/L_x$ as

$$t^{(y)}(\Delta y)\phi_{n,k}^{\mathrm{P}} = \phi_{n,k'}^{\mathrm{P}} \quad \textit{with} \quad k' = k + \frac{\Delta y}{\ell_B^2} = k + \frac{2\pi}{L_x}. \tag{3.20}$$

Further,

$$R\phi_{n,k}^{\mathrm{P}} = (-1)^n \phi_{n,-k}^{\mathrm{P}}. \tag{3.21}$$

As usual we denote by $L^2(S)$ the set of functions f on the rectangular box S such that

$$\int_S dxdy\ |f(x,y)|^2 = \int_{-L_x/2}^{L_x/2} dx \int_{-L_y/2}^{L_y/2} dy\ |f(x,y)|^2 < \infty. \tag{3.22}$$

Further we define the associate inner product (f,g) as

$$(f,g) = \int_S dxdy\ [f(x,y)]^* g(x,y) \tag{3.23}$$

for $f, g \in L^2(S)$.

Lemma 3.2 *Let f, g be functions on $\mathbf{R}^2$ such that $f, g \in L^2(S)$, and that f, g satisfy the boundary conditions (3.9). Then*

$$(f,g) = \int_{-L_x/2}^{L_x/2} dx \int_{-L_y/2+y_0}^{L_y/2+y_0} dy\ [f(x,y)]^* g(x,y) \tag{3.24}$$

for any $y_0 \in \mathbf{R}$.

Proof: By the periodic boundary condition $f(x,y) = t^{(x)}(L_x)f(x,y)$, the function f can be expanded in Fourier series as

$$f(x,y) = L_x^{-1/2}\sum_k e^{ikx}\hat{f}(k,y). \tag{3.25}$$

Further, since

$$
\begin{aligned}
f(x,y) = t^{(y)}(L_y) f(x,y) &= L_x^{-1/2} \sum_k e^{i(k+K)x} \hat{f}(k, y-L_y) \\
&= L_x^{-1/2} \sum_k e^{ikx} \hat{f}(k-K, y-L_y),
\end{aligned}
\tag{3.26}
$$

one has

$$
\hat{f}(k,y) = \hat{f}(k-K, y-L_y). \tag{3.27}
$$

Using this relation repeatedly, the function f of (3.25) can be rewritten as

$$
f(x,y) = \sum_{\{k=2\pi n/L_x | -M/2+1 \le n \le M/2\}} L_x^{-1/2} \sum_{\ell=-\infty}^{+\infty} e^{i(k+\ell K)x} \hat{f}(k, y-\ell L_y). \tag{3.28}
$$

By the help of this expression, one has

$$
\begin{aligned}
(f,g) &= \int_{-L_x/2}^{L_x/2} dx \int_{-L_y/2}^{L_y/2} dy \ \left[f(x,y)\right]^* g(x,y) \\
&= \sum_{\{k=2\pi n/L_x | -M/2+1 \le n \le M/2\}} \sum_{\ell=-\infty}^{+\infty} \int_{-L_y/2}^{L_y/2} dy \ \left[\hat{f}(k, y-\ell L_y)\right]^* \hat{g}(k, y-\ell L_y) \\
&= \sum_{\{k=2\pi n/L_x | -M/2+1 \le n \le M/2\}} \int_{-\infty}^{+\infty} dy \ \left[\hat{f}(k,y)\right]^* \hat{g}(k,y) \\
&= \sum_{\{k=2\pi n/L_x | -M/2+1 \le n \le M/2\}} \sum_{\ell=-\infty}^{+\infty} \int_{-L_y/2+y_0}^{L_y/2+y_0} dy \ \left[\hat{f}(k, y-\ell L_y)\right]^* \hat{g}(k, y-\ell L_y) \\
&= \int_{-L_x/2}^{L_x/2} dx \int_{-L_y/2+y_0}^{L_y/2+y_0} dy \ \left[f(x,y)\right]^* g(x,y).
\end{aligned}
\tag{3.29}
$$

■

Let us see that the set of the eigenvectors $\{\phi_{n,k}^{\mathrm{P}}\}$ of (3.17) forms an orthonormal complete system. From (3.29) in Lemma 3.2, one has

$$
\left(\phi_{n',k'}^{\mathrm{P}}, \phi_{n,k}^{\mathrm{P}}\right) = \int_{-\infty}^{+\infty} dy \ v_{n',k}^*(y) v_{n,k}(y) \delta_{k,k'} = \delta_{n,n'} \delta_{k,k'}. \tag{3.30}
$$

Here $\delta_{k,k'}$ is the Kronecker delta. To show the completeness, consider a function f satisfying the boundary conditions (3.9). In the same way,

$$
\left(\phi_{n,k}^{\mathrm{P}}, f\right) = \int_{-\infty}^{+\infty} dy \ v_{n,k}^*(y) \hat{f}(k,y). \tag{3.31}
$$

This implies that the function f must be zero if the inner product $\left(\phi_{n,k}^{\mathrm{P}}, f\right)$ is vanishing for all the vectors $\phi_{n,k}^{\mathrm{P}}$.

3.2 The Landau Hamiltonian in an electric field

Next we consider a single electron in magnetic and electric fields in the rectangular box $S = [-L_x/2, L_x/2] \times [-L_y/2, L_y/2]$. The Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2m_e} [\mathbf{p} + e\mathbf{A}(\mathbf{r})]^2 + eFyP_{\rm bulk} - \frac{m_e}{2} \left(\frac{F}{B}\right)^2. \tag{3.32}$$

We take the vector potential as

$$e\mathbf{A}(\mathbf{r}) = (-eBy + A_0, 0, 0) \tag{3.33}$$

which gives the constant magnetic field $\mathbf{B} = (0, 0, B)$ perpendicular to the x-y plane. We also applied the constant electric field $\mathbf{F} = (0, F, 0)$ oriented along the y axis. We have introduced the projection operator $P_{\rm bulk}$ so that the electrons near the boundaries $y = \pm L_y/2$ do not feel the infinitely strong electric field at the boundaries. The precise definition of $P_{\rm bulk}$ is given as follows: We define a projection operator $P(k)$ onto the Fourier component with a wavenumber k for a function $f \in L^2(S)$ as

$$P(k)f(x, y) = L_x^{-1/2} e^{ikx} \hat{f}(k, y) \tag{3.34}$$

with the Fourier coefficient

$$\hat{f}(k, y) = L_x^{-1/2} \int_{-L_x/2}^{L_x/2} dx\, e^{-ikx} f(x, y). \tag{3.35}$$

For an interval I we define a projection operator as

$$P(I) := \sum_{k \in \mathcal{F}(I)} P(k) \tag{3.36}$$

with

$$\mathcal{F}(I) := \left\{ k = \frac{2\pi n}{L_x} \,\middle|\, n \in \mathbf{Z} \quad \text{and} \quad \frac{\hbar k}{eB} \in I \right\}. \tag{3.37}$$

We define $P_{\rm bulk}$ as

$$P_{\rm bulk} := P(I_{\rm bulk}) \tag{3.38}$$

with the interval

$$I_{\rm bulk} := \bigcup_{n=-\infty}^{+\infty} [-L_y/2 + \delta + nL_y, L_y/2 - \delta + nL_y] \tag{3.39}$$

with a positive number δ. We choose δ satisfying $\delta/L_y \to 0$ as $L_y \to +\infty$ so that the effect of the projection $(1 - P_{\rm bulk})$ at the boundaries is negligible[11] in the thermodynamic limit $L_y \to +\infty$. Here we stress that the operator $eFyP_{\rm bulk}$ in the Hamiltonian $\mathcal{H}$ of (3.32) is self-adjoint because y and $P_{\rm bulk}$ commute with each other from their definitions.

Next let us show the locality of $(1 - P_{\rm bulk})$. Namely the operator is vanishing on the bulk region which is at a distance from the boundaries. Therefore the electric field is constant on the bulk region.

[11] Unfortunately we cannot prove this claim.

Let $\Phi^{(N)}$ be an N electrons wavefunction satisfying

$$\frac{1}{N}\left\langle \Phi^{(N)}, H_{\omega,0}^{(N)}\Phi^{(N)}\right\rangle < \tilde{\mathcal{E}} < \infty, \tag{3.40}$$

where the positive constant $\tilde{\mathcal{E}}$ is independent of the linear dimensions L_x, L_y and of the number N of electrons, and $H_{\omega,0}^{(N)}$ is the Hamiltonian $H_\omega^{(N)}$ of (2.1) with $A_0 = 0$ and $F = 0$. Then we have the following bound:

$$\left|\left\langle \Phi^{(N)}, \chi_{\rm bulk}(y_j)(1-P_{{\rm bulk},j})\Phi^{(N)}\right\rangle\right| \le \frac{2(\tilde{\mathcal{E}}+\|V_\omega\|)}{\hbar\omega_c}\left(\frac{\ell_B}{\delta}\right)^2, \tag{3.41}$$

where $\chi_{\rm bulk}$ is a characteristic function given by

$$\chi_{\rm bulk}(y) := \begin{cases} 1, & \text{for } y \in [-L_y/2+2\delta, L_y/2-2\delta]; \\ 0, & \text{otherwise.} \end{cases} \tag{3.42}$$

Since we can choose $\delta \to \infty$ as $L_y \to \infty$, this bound clearly implies the locality of $(1-P_{{\rm bulk},j})$. Let us prove the bound. Note that

$$(eB\delta)^2\chi_{\rm bulk}(y_j)(1-P_{{\rm bulk},j}) \le \chi_{\rm bulk}(y_j)(1-P_{{\rm bulk},j})(p_{x,j}-eBy_j)^2 \tag{3.43}$$

from the definitions. Using this inequality and the assumption (3.40), we have

$$\begin{aligned} & (eB\delta)^2\left\langle \Phi^{(N)}, \chi_{\rm bulk}(y_j)(1-P_{{\rm bulk},j})\Phi^{(N)}\right\rangle \\ \le\ & \left\langle \Phi^{(N)}, \chi_{\rm bulk}(y_j)(1-P_{{\rm bulk},j})(p_{x,j}-eBy_j)^2\Phi^{(N)}\right\rangle \\ \le\ & \left\langle \Phi^{(N)}, (p_{x,j}-eBy_j)^2\Phi^{(N)}\right\rangle \\ \le\ & 2m_e\left(\frac{1}{N}\left\langle \Phi^{(N)}, H_{\omega,0}^{(N)}\Phi^{(N)}\right\rangle + \|V_\omega\|\right) \le 2m_e\left(\tilde{\mathcal{E}}+\|V_\omega\|\right), \end{aligned} \tag{3.44}$$

where we have used the positivity of the electron-electron interaction $U^{(2)}$ for getting the third inequality. This is nothing but the desired bound (3.41).

For the convenience of the following calculations, we choose $A_0 = m_eF/B$. Then the Hamiltonian $\mathcal{H}$ of (3.32) becomes

$$\begin{aligned} \mathcal{H} &= \frac{1}{2m_e}\left[(p_x-eBy+A_0)^2+p_y^2\right] + eFyP_{\rm bulk} - \frac{m_e}{2}\left(\frac{F}{B}\right)^2 \\ &= \frac{1}{2m_e}\left[(p_x-eBy)^2+p_y^2\right] + \frac{1}{2m_e}[2A_0(p_x-eBy)+A_0^2] + eFyP_{\rm bulk} - \frac{m_e}{2}\left(\frac{F}{B}\right)^2 \\ &= \frac{1}{2m_e}\left[(p_x-eBy)^2+p_y^2\right] + \frac{F}{B}p_xP_{\rm bulk} + \frac{F}{B}(p_x-eBy)\left(1-P_{\rm bulk}\right), \end{aligned} \tag{3.45}$$

and the velocity operator v_x is given by

$$m_ev_x = p_x - eBy + \frac{m_eF}{B}. \tag{3.46}$$

In the following we will treat the second and the third terms in the last line of (3.45) as a perturbation.

4 The relation between the present "electric potential" and the standard time-dependent vector potential

In the same setting as in Section 2, we introduce the standard time-dependent vector potential $\mathbf{A}(t)$ instead of the regularized electric potential so that the vector potential yields the constant electric field on the whole torus. Since the electric potential gives the constant electric field except for the neighborhood of the boundaries as we have seen in Section 3.2, we can expect that these two different potentials yield the same transport properties in the large volume limit. In this section, we shall discuss this issue.

The time-dependent Schrödinger equation with the vector potential is given by

$$i\hbar\frac{\partial}{\partial t}\Phi^{(N)}(t) = H_\omega^{(N)}(t)\Phi^{(N)}(t) \tag{4.1}$$

with the time-dependent Hamiltonian

$$H_\omega^{(N)}(t) = \sum_{j=1}^{N}\frac{1}{2m_e}\left\{(p_{x,j}-eBy_j)^2+[p_{y,j}+eA(t)]^2\right\}+\tilde{U}_\omega^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N) \tag{4.2}$$

with the vector potential $\mathbf{A}(t)=(0,A(t),0)$ with

$$A(t) = -Fte^{\eta t} \quad \text{for} \quad -\infty < t \le 0, \tag{4.3}$$

and with the potentials

$$\tilde{U}_\omega^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N) = \sum_{j=1}^{N} V_\omega(\mathbf{r}_j)+\sum_{i<j}U^{(2)}(\mathbf{r}_i-\mathbf{r}_j). \tag{4.4}$$

Here η is a small positive parameter switching the corresponding electric field adiabatically.

In order to show the equivalence between the two systems, we introduce a unitary transformation $\Phi^{(N)}(t) = G^{(N)}(t)\Psi^{(N)}(t)$ with

$$G^{(N)}(t) = \exp\left[-\frac{i}{\hbar}\sum_{j=1}^{N} ey_jP_{\mathrm{bulk},j}\tilde{\chi}_{\mathrm{bulk}}(y_j)A(t)\right], \tag{4.5}$$

where we have introduced the function $\tilde{\chi}_{\mathrm{bulk}}$ so that the wavefunctions satisfy the periodic boundary conditions. We take the function $\tilde{\chi}_{\mathrm{bulk}}$ to be an infinitely differentiable function satisfying

$$\tilde{\chi}_{\mathrm{bulk}}(y) = \begin{cases} 1, & \text{for } y \in [-L_y/2+2\delta/3, L_y/2-2\delta/3]; \\ 0, & \text{for } y \in [-L_y/2, -L_y/2+\delta/3]\cup[L_y/2-\delta/3, L_y/2]. \end{cases} \tag{4.6}$$

Namely it is equal to the identity on the bulk region and vanishing near the boundaries. The effect of $\tilde{\chi}_{\mathrm{bulk}}$ to the conductance is negligible for the large volume[12] by the locality of P_{bulk}.

[12] One can prove the statement by using the method in the present paper, although we do not give the proof here.

Note that

$$i\hbar\frac{\partial}{\partial t}\Phi^{(N)}(t)=G^{(N)}(t)i\hbar\frac{\partial}{\partial t}\Psi^{(N)}(t)+\left[i\hbar\frac{\partial}{\partial t}G^{(N)}(t)\right]\Psi^{(N)}(t), \tag{4.7}$$

and

$$\left[G^{(N)}(t)\right]^{*}i\hbar\frac{\partial}{\partial t}G^{(N)}(t)=-\sum_{j=1}^{N}ey_jF(1+\eta t)e^{\eta t}P_{\mathrm{bulk},j}\tilde{\chi}_{\mathrm{bulk}}(y_j). \tag{4.8}$$

Since η is an infinitesimally small parameter, the right-hand side leads to the regularized electric potential of the present paper. We also have

$$\begin{aligned}&\left[G^{(N)}(t)\right]^{*}[p_{y,j}+eA(t)]G^{(N)}(t)\\ =\ & p_{y,j}+eA(t)\left[1-P_{\mathrm{bulk},j}\tilde{\chi}_{\mathrm{bulk}}(y_j)\right]-eA(t)y_jP_{\mathrm{bulk},j}\frac{\partial}{\partial y_j}\tilde{\chi}_{\mathrm{bulk}}(y_j).\end{aligned} \tag{4.9}$$

The second and third terms in the right-hand side are vanishing in the large volume limit for getting the conductance. This statement can be proved in the same way as in the present paper. From these observations, we obtain

$$i\hbar\frac{\partial}{\partial t}\Psi^{(N)}(t)=\left[H_{\omega}^{(N)}+\Delta\tilde{U}_{\omega}^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N;t)\right]\Psi^{(N)}(t) \tag{4.10}$$

with the Hamiltonian $H_{\omega}^{(N)}$ of (2.1) and

$$\Delta\tilde{U}_{\omega}^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N;t)=\left[G^{(N)}(t)\right]^{*}\tilde{U}_{\omega}^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)G^{(N)}(t)-\tilde{U}_{\omega}^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N). \tag{4.11}$$

Here we have dropped some terms which do not contribute to the conductance in the large volume limit. If we can drop the potential $\Delta\tilde{U}_{\omega}^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N;t)$, then we get the desired result, i.e., the unitary equivalence between the two systems with the different potentials in the large volume limit. Unfortunately we can not dropp the potential. But we can expect that the contribution to conductance is of order of δ/L_y which is vanishing in the limit $L_y\to\infty$.

Let us estimate the correction from $\Delta\tilde{U}_{\omega}^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N;t)$ to the conductance. Note that the unitary operator $G^{(N)}(t)$ can be rewritten as

$$G^{(N)}(t)=\exp\left[-\frac{i}{\hbar}\sum_{j=1}^{N}ey_jA(t)\tilde{\chi}_{\mathrm{bulk}}(y_j)\right]G_{\mathrm{edge}}^{(N)}(t) \tag{4.12}$$

with

$$G_{\mathrm{edge}}^{(N)}(t)=\exp\left[\frac{i}{\hbar}\sum_{j=1}^{N}ey_jA(t)(1-P_{\mathrm{bulk},j})\tilde{\chi}_{\mathrm{bulk}}(y_j)\right]. \tag{4.13}$$

Using this, we have

$$\left[G^{(N)}(t)\right]^{*}\tilde{U}_{\omega}^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)G^{(N)}(t)=\left[G_{\mathrm{edge}}^{(N)}(t)\right]^{*}\tilde{U}_{\omega}^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)G_{\mathrm{edge}}^{(N)}(t). \tag{4.14}$$

Immediately, we get

$$\Delta\tilde{U}_{\omega}^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N;t)=\left[G_{\mathrm{edge}}^{(N)}(t)\right]^{*}\tilde{U}_{\omega}^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)G_{\mathrm{edge}}^{(N)}(t)-\tilde{U}_{\omega}^{(N)}(\mathbf{r}_1,\ldots,\mathbf{r}_N). \tag{4.15}$$

This implies, owing to the definition of $G_{\rm edge}^{(N)}(t)$ and the locality of $P_{\rm bulk}$, that, if the potential $\tilde{U}_\omega^{(N)}(\mathbf{r}_1,\dots,\mathbf{r}_N)$ is vanishing near the boundaries, then $\Delta\tilde{U}_\omega^{(N)}(\mathbf{r}_1,\dots,\mathbf{r}_N;t)$ is almost vanishing on the whole torus. In fact, if we take the potential

$$\tilde{U}_\omega^{(N)}(\mathbf{r}_1,\dots,\mathbf{r}_N)=\sum_{j=1}^N P_{{\rm bulk},j}V_\omega(\mathbf{r}_j)P_{{\rm bulk},j}+\sum_{i<j}P_{{\rm bulk},i}P_{{\rm bulk},j}U^{(2)}(\mathbf{r}_i-\mathbf{r}_j)P_{{\rm bulk},i}P_{{\rm bulk},j} \quad (4.16)$$

instead of (4.4), then the potential difference $\Delta\tilde{U}_\omega^{(N)}(\mathbf{r}_1,\dots,\mathbf{r}_N;t)$ is exactly equal to zero. Thus $\Delta\tilde{U}_\omega^{(N)}(\mathbf{r}_1,\dots,\mathbf{r}_N;t)$ is vanishing on the bulk region, and the correction to the conductance is expected to be of order of δ/L_y. Unfortunately we could not estimate the correction in a mathematically rigorous sense.

5 Proofs of the main theorems

In this section, we calculate the conductivities which are derived as the linear response coefficients for the weak electric field. For this purpose, we use the Rayleigh-Schrödinger perturbation theories. Our goal is to give proofs of our main Theorems 2.1 and 2.2. Namely we show that the conductivities (conductance) so obtained satisfy the bounds in the theorems. For the convenience of readers, in Appendices A-H we give technical estimates and calculations of matrix elements appeared in our representation of the conductivities.

By choosing $A_0=m_eF/B$ as in (3.45) in Section 3.2, the N electrons Hamiltonian $H_\omega^{(N)}$ of (2.1) which we mainly treat in this paper can be rewritten as

$$H_\omega^{(N)}=H_{\omega,0}^{(N)}+\lambda\tilde{H}^{(N)} \quad (5.1)$$

with

$$H_{\omega,0}^{(N)}=\sum_{j=1}^N\left[\frac{1}{2m_e}\left(p_{x,j}-eBy_j\right)^2+\frac{1}{2m_e}p_{y,j}^2+V_\omega(\mathbf{r}_j)\right]+U^{(N)}(\mathbf{r}_1,\dots,\mathbf{r}_N) \quad (5.2)$$

and

$$\tilde{H}^{(N)}=\sum_{j=1}^N\left[p_{x,j}P_{{\rm bulk},j}+\left(p_{x,j}-eBy_j\right)\left(1-P_{{\rm bulk},j}\right)\right]. \quad (5.3)$$

Here $\lambda=F/B$ is a real parameter, $U^{(N)}$ is written in a sum of two-body interactions as

$$U^{(N)}(\mathbf{r}_1,\dots,\mathbf{r}_N)=\sum_{1\le i<j\le N}U^{(2)}(x_i-x_j,y_i-y_j), \quad (5.4)$$

and we have dropped the constant term.

We treat $\tilde{H}^{(N)}$ as a perturbation, and apply the Rayleigh-Schrödinger perturbation theory to the eigenvalue problem of the Hamiltonian $H_\omega^{(N)}$ of (5.1) for getting a "ground state" eigenvector of $H_\omega^{(N)}$.

5.1 The Hall and diagonal conductivities

5.1.1 Non-degenerate ground state

Consider first the case when the ground state $\Phi_{\omega,0}^{(N)}$ of the unperturbed Hamiltonian $H_{\omega,0}^{(N)}$ is non-degenerate. Let $\tilde{\Phi}_{\omega,0}^{(N)}$ be the corresponding normalized ground state eigenvector of the full Hamiltonian $H_\omega^{(N)}$. Since the electric field F is assumed to be sufficiently weak, the ground state eigenvector $\tilde{\Phi}_{\omega,0}^{(N)}$ is unique. Then the current density averaged over the random potentials $\Omega(\omega)$ at zero temperature is given by

$$j_s = -\frac{e}{L_x L_y}\mathbf{E}_\omega\left[\left\langle \tilde{\Phi}_{\omega,0}^{(N)}, v_{\mathrm{tot},s}\tilde{\Phi}_{\omega,0}^{(N)}\right\rangle\right] \quad \text{for } s = x, y, \tag{5.5}$$

where the velocity operator $v_{\mathrm{tot},s}$ for the N electrons is given by

$$v_{\mathrm{tot},x} := \sum_{j=1}^N v_{x,j} = N\frac{F}{B} + \frac{1}{m_e}\sum_{j=1}^N (p_{x,j} - eBy_j), \tag{5.6}$$

and

$$v_{\mathrm{tot},y} := \frac{1}{m_e}\sum_{j=1}^N p_{y,j}. \tag{5.7}$$

We rewrite the current density $\mathbf{j} = (j_x, j_y)$ as

$$j_s = \begin{cases} -\dfrac{e^2}{h}\nu F + \Delta j_x & \text{for } s = x; \\ \Delta j_y & \text{for } s = y \end{cases} \tag{5.8}$$

with

$$\Delta j_s := -\frac{e}{m_e L_x L_y}\sum_{j=1}^N \mathbf{E}_\omega\left[\left\langle \tilde{\Phi}_{\omega,0}^{(N)}, \pi_{s,j}\tilde{\Phi}_{\omega,0}^{(N)}\right\rangle\right], \tag{5.9}$$

where

$$\pi_s := \begin{cases} p_x - eBy & \text{for } s = x\ ; \\ p_y & \text{for } s = y, \end{cases} \tag{5.10}$$

and ν is the filling factor for the Landau level. Namely $N = \nu M$ with the number $M = eBL_xL_y/h$ which is equal to the number of states in a single Landau level of the non-interacting Landau Hamiltonian with no disorder.

From the standard formula of the perturbation theory, the ground state eigenvector $\tilde{\Phi}_{\omega,0}^{(N)}$ of $H_\omega^{(N)}$ is expanded as

$$\tilde{\Phi}_{\omega,0}^{(N)} = \Phi_{\omega,0}^{(N)} + \lambda\sum_{\ell\neq 0}\Phi_{\omega,\ell}^{(N)}\frac{\left\langle \Phi_{\omega,\ell}^{(N)}, \tilde{H}^{(N)}\Phi_{\omega,0}^{(N)}\right\rangle}{E_{\omega,0}^{(N)} - E_{\omega,\ell}^{(N)}} + \mathcal{O}(\lambda^2) \tag{5.11}$$

in powers of λ. Here $\Phi_{\omega,\ell}^{(N)}$ are the orthonormal eigenvectors of the unperturbed Hamiltonian $H_{\omega,0}^{(N)}$ with the energy eigenvalues $E_{\omega,\ell}^{(N)}$. For the detail, see Appendix A.1. Using this

expansion (5.11), we have

$$\begin{aligned}\sum_{j=1}^{N}\left\langle\tilde{\Phi}_{\omega,0}^{(N)},\pi_{s,j}\tilde{\Phi}_{\omega,0}^{(N)}\right\rangle &= \sum_{j=1}^{N}\left\langle\Phi_{\omega,0}^{(N)},\pi_{s,j}\Phi_{\omega,0}^{(N)}\right\rangle \\ &+ 2\lambda\sum_{\ell\neq 0}\sum_{j=1}^{N}\mathrm{Re}\left[\frac{\left\langle\Phi_{\omega,\ell}^{(N)},\tilde{H}^{(N)}\Phi_{\omega,0}^{(N)}\right\rangle}{E_{\omega,0}^{(N)}-E_{\omega,\ell}^{(N)}}\left\langle\Phi_{\omega,0}^{(N)},\pi_{s,j}\Phi_{\omega,\ell}^{(N)}\right\rangle\right]+\mathcal{O}(\lambda^2),\end{aligned} \tag{5.12}$$

where Re stands for a real part. By Lemma B.6 in Appendix B, the average of the first sum is vanishing as

$$\sum_{j=1}^{N}\mathbf{E}_\omega\left[\left\langle\Phi_{\omega,0}^{(N)},\pi_{s,j}\Phi_{\omega,0}^{(N)}\right\rangle\right]=0. \tag{5.13}$$

Substituting (5.12) and (5.13) into the right-hand side of (5.9), we have

$$\Delta j_s=\Delta j_s^{(1)}+\mathcal{O}\left((F/B)^2\right) \tag{5.14}$$

with

$$\Delta j_s^{(1)}:=-\frac{2e^2}{h}\nu F\ \mathrm{Re}\mathbf{E}_\omega\left[\mathcal{M}_s\right], \tag{5.15}$$

where

$$\mathcal{M}_s:=\frac{1}{m_eN}\sum_{j=1}^{N}\left\langle\Phi_{\omega,0}^{(N)},\pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}\tilde{H}^{(N)}\Phi_{\omega,0}^{(N)}\right\rangle. \tag{5.16}$$

Here $G_\omega^{(N)}$ is the orthogonal projection onto the ground state $\Phi_{\omega,0}^{(N)}$. From (5.8), (5.9), (5.14) and (5.15), the Hall and diagonal conductivities can be written as

$$\sigma_{xy}:=\lim_{F\to 0}\frac{j_x}{F}=-\frac{e^2}{h}\nu-\frac{2e^2}{h}\nu\mathrm{Re}\mathbf{E}_\omega\left[\mathcal{M}_x\right] \tag{5.17}$$

and

$$\sigma_{yy}:=\lim_{F\to 0}\frac{j_y}{F}=-\frac{2e^2}{h}\nu\mathrm{Re}\mathbf{E}_\omega\left[\mathcal{M}_y\right], \tag{5.18}$$

respectively.

5.1.2 Degenerate "ground state"

Next consider the case when the "ground state" of the Hamiltonian $H_{\omega,0}^{(N)}$ is q-fold degenerate. Let $\Phi_{\omega,(0,\mu)}^{(N)}$ be the "ground state" eigenvectors with the energy eigenvalue $E_{\omega,(0,\mu)}^{(N)}$ for $\mu=1,2,\ldots,q$. We take $\left\{\Phi_{\omega,(0,\mu)}^{(N)}\right\}$ to be an orthonormal system. In this case, the current desity is given by

$$j_s:=-\frac{e}{L_xL_y}\mathbf{E}_\omega\left[\frac{1}{q}\sum_{\mu=1}^{q}\left\langle\tilde{\Phi}_{\omega,(0,\mu)}^{(N)},v_{\mathrm{tot},s}\tilde{\Phi}_{\omega,(0,\mu)}^{(N)}\right\rangle\right], \tag{5.19}$$

where $\tilde{\Phi}^{(N)}_{\omega,(0,\mu)}$ are the corresponding normalized ground state eigenvectors of the Hamiltonian $H^{(N)}_{\omega}$, with the corresponding energy eigenvalues $\tilde{E}^{(N)}_{\omega,(0,\mu)}$. Similarly to the non-degenerate case, the corrections for the current density $\mathbf{j}$ are given by

$$\Delta j_s := -\frac{e}{m_e L_x L_y}\mathbf{E}_\omega\left[\frac{1}{q}\sum_{\mu=1}^{q}\left\langle \tilde{\Phi}^{(N)}_{\omega,(0,\mu)}, \sum_{j=1}^{N}\pi_{s,j}\tilde{\Phi}^{(N)}_{\omega,(0,\mu)}\right\rangle\right]. \tag{5.20}$$

The "ground state" eigenvectors $\tilde{\Phi}^{(N)}_{\omega,(0,\mu)}$ are expanded as

$$\tilde{\Phi}^{(N)}_{\omega,(0,\mu)} = \Phi^{(N,0)}_{\omega,(0,\mu)} + \lambda\sum_{\ell\neq 0}\Phi^{(N)}_{\omega,\ell}\frac{1}{E^{(N)}_{\omega,(0,\mu)} - E^{(N)}_{\omega,\ell}}\left\langle \Phi^{(N)}_{\omega,\ell}, \tilde{H}^{(N)}\Phi^{(N,0)}_{\omega,(0,\mu)}\right\rangle + \cdots \tag{5.21}$$

by using the degenerate perturbation theory. Here $\Phi^{(N,0)}_{\omega,(0,\mu)}$ are orthonormal vectors which span the sector spanned by the "ground states" eigenvectors $\Phi^{(N)}_{\omega,(0,\mu)}$ of the unperturbed Hamiltonian $H^{(N)}_{\omega,0}$. For the detail of the degenerate perturbation theory, see Appendix A.2. Using this expansion, we have

$$\begin{aligned}
&\left\langle \tilde{\Phi}^{(N)}_{\omega,(0,\mu)}, \sum_{j=1}^{N}\pi_{s,j}\tilde{\Phi}^{(N)}_{\omega,(0,\mu)}\right\rangle \\
&= \left\langle \Phi^{(N,0)}_{\omega,(0,\mu)}, \sum_{j=1}^{N}\pi_{s,j}\Phi^{(N,0)}_{\omega,(0,\mu)}\right\rangle + 2\lambda\mathrm{Re}\left[\left\langle \Phi^{(N,0)}_{\omega,(0,\mu)}, \sum_{j=1}^{N}\pi_{s,j}\frac{1-G^{(N)}_{\omega}}{E^{(N)}_{\omega,(0,\mu)} - H^{(N)}_{\omega,0}}\tilde{H}^{(N)}\Phi^{(N)}_{\omega,(0,\mu)}\right\rangle\right] \\
&\qquad +\mathcal{O}(\lambda^2).
\end{aligned} \tag{5.22}$$

Here $G^{(N)}_{\omega}$ is the orthogonal projection onto the sector of the degenerate "ground state" whose space is spanned by the q energy eigenvectors $\Phi^{(N)}_{\omega,(0,\mu)}$, $\mu = 1, 2, \ldots, q$. Substituting (5.22) into (5.20), we obtain

$$\begin{aligned}
\Delta j_s &= -\frac{e}{m_e L_x L_y}\mathbf{E}_\omega\left[\frac{1}{q}\sum_{\mu=1}^{q}\left\langle \Phi^{(N,0)}_{\omega,(0,\mu)}, \sum_{j=1}^{N}\pi_{s,j}\Phi^{(N,0)}_{\omega,(0,\mu)}\right\rangle\right] \\
&- \frac{2e\lambda}{m_e L_x L_y}\mathrm{Re}\mathbf{E}_\omega\left[\frac{1}{q}\sum_{\mu=1}^{q}\left\langle \Phi^{(N,0)}_{\omega,(0,\mu)}, \sum_{j=1}^{N}\pi_{s,j}\frac{1-G^{(N)}_{\omega}}{E^{(N)}_{\omega,(0,\mu)} - H^{(N)}_{\omega,0}}\tilde{H}^{(N)}\Phi^{(N,0)}_{\omega,(0,\mu)}\right\rangle\right] + \mathcal{O}(\lambda^2) \\
&= -\frac{e}{m_e L_x L_y}\mathbf{E}_\omega\left[\frac{1}{q}\sum_{\mu=1}^{q}\left\langle \Phi^{(N)}_{\omega,(0,\mu)}, \sum_{j=1}^{N}\pi_{s,j}\Phi^{(N)}_{\omega,(0,\mu)}\right\rangle\right] \\
&- \frac{2e\lambda}{m_e L_x L_y}\mathrm{Re}\mathbf{E}_\omega\left[\frac{1}{q}\sum_{\mu=1}^{q}\left\langle \Phi^{(N)}_{\omega,(0,\mu)}, \sum_{j=1}^{N}\pi_{s,j}\frac{1-G^{(N)}_{\omega}}{E^{(N)}_{\omega,(0,\mu)} - H^{(N)}_{\omega,0}}\tilde{H}^{(N)}\Phi^{(N)}_{\omega,(0,\mu)}\right\rangle\right] + \mathcal{O}(\lambda^2).
\end{aligned} \tag{5.23}$$

Since the first term in the right-hand side of the second equality is vanishing owing to Lemma B.6, we obatin

$$\Delta j_s = \Delta j_s^{(1)} + \mathcal{O}(F^2) \tag{5.24}$$

with

$$\Delta j_s^{(1)} := -\frac{2e^2}{h}\nu F \mathrm{Re}\mathbf{E}_\omega\left[\mathcal{M}_s\right] \tag{5.25}$$

and

$$\mathcal{M}_s := \frac{1}{m_e N}\frac{1}{q}\sum_{\mu=1}^{q}\left\langle \Phi_{\omega,(0,\mu)}^{(N)}, \sum_{j=1}^{N}\pi_{s,j}\frac{1-G_\omega^{(N)}}{E_{\omega,(0,\mu)}^{(N)}-H_{\omega,0}^{(N)}}\tilde{H}^{(N)}\Phi_{\omega,(0,\mu)}^{(N)}\right\rangle. \tag{5.26}$$

In consequence, we have the expressions of the conductivities as

$$\sigma_{xy} := \lim_{F\to 0}\frac{j_x}{F} = -\frac{e^2}{h}\nu - \frac{2e^2}{h}\nu \mathrm{Re}\mathbf{E}_\omega\left[\mathcal{M}_x\right] \tag{5.27}$$

and

$$\sigma_{yy} := \lim_{F\to 0}\frac{j_y}{F} = -\frac{2e^2}{h}\nu \mathrm{Re}\mathbf{E}_\omega\left[\mathcal{M}_y\right] \tag{5.28}$$

with the above $\mathcal{M}_s$. These have the same forms as (5.17) and (5.18) in the non-degenerate case.

5.2 Estimate of $\mathbf{E}_\omega[\mathcal{M}_s]$

From the expressions of the conductivities (5.17), (5.18), (5.27) and (5.28), we want to estimate $\mathbf{E}_\omega[\mathcal{M}_s]$, in order to prove Theorems 2.1 and 2.2. In the following, we treat only the non-degenerate case because one can treat the degenerate case in the same way.

We define two projection operators P_{in} and P_{out} as

$$P_{\mathrm{in}} := P(I_{\mathrm{in}}), \quad \text{and} \quad P_{\mathrm{out}} := P(I_{\mathrm{out}}) \tag{5.29}$$

with the intervals

$$I_{\mathrm{in}} = [-L_y/2+\delta, L_y/2-\delta], \quad \text{and} \quad I_{\mathrm{out}} = I_{\mathrm{bulk}}\backslash I_{\mathrm{in}}. \tag{5.30}$$

Clearly we have $P_{\mathrm{bulk}} = P_{\mathrm{in}} + P_{\mathrm{out}}$ from the definition (3.38) of P_{bulk} with (3.39). We write $\mathcal{M}_s$ of (5.16) as

$$\mathcal{M}_s = \mathcal{M}_{s,\mathrm{in}} + \mathcal{M}_{s,\mathrm{out}} + \mathcal{M}_{s,\mathrm{edge}}, \tag{5.31}$$

where

$$\mathcal{M}_{s,\mathrm{in}} := \frac{1}{m_e N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}P_{\mathrm{in},i}p_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle, \tag{5.32}$$

$$\mathcal{M}_{s,\mathrm{out}} := \frac{1}{m_e N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}P_{\mathrm{out},i}p_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle, \tag{5.33}$$

and

$$\mathcal{M}_{s,\mathrm{edge}} := \frac{1}{m_e N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}(1-P_{\mathrm{bulk},i})(p_{x,i}-eBy_i)\Phi_{\omega,0}^{(N)}\right\rangle. \tag{5.34}$$

Let us sketch the idea of the proofs of Theorems 2.1 and 2.2. Since one can expect that the contributions of $\mathcal{M}_{s,\mathrm{out}}$ and $\mathcal{M}_{s,\mathrm{edge}}$ become small for a large volume, we explain

the idea only for $\mathcal{M}_{s,\mathrm{in}}$. Consider the random average of the matrix element in (5.32). It is written as

$$\begin{aligned}
&\mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}P_{\mathrm{in},i}p_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right] \\
&= \sum_k \hbar k \mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}P_i(k)\Phi_{\omega,0}^{(N)}\right\rangle\right] \qquad (5.35)
\end{aligned}$$

by using the projection operator $P(k)$ onto the Fourier component with the wavenumber k. We introduce a transformation consisting of a reflection and a magnetic translation as

$$x \to -x \quad , \quad y \to 2y_k - y \qquad (5.36)$$

with $y_k = \hbar k/(eB)$. In particular, $y = y_k$ is the fixed point for the second part of the transformation. This yields that the wavenumber k also is the fixed point in the space of the wavenumbers. Using the transformation, we have

$$\begin{aligned}
&\hbar k \mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}P_i(k)\Phi_{\omega,0}^{(N)}\right\rangle\right] \\
&= -\hbar k \mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}P_i(k)\Phi_{\omega,0}^{(N)}\right\rangle\right] \\
&+ \ (\text{corrections from the boundaries } y = \pm L_y/2) \qquad (5.37)
\end{aligned}$$

for the summand with k in the right-hand side of (5.35). From these observations, we conclude that the contributions of $\mathbf{E}_\omega[\mathcal{M}_{s,\mathrm{in}}]$ for the conductivities are small if the corrections from the boundaries give small contributions for a large volume. In fact, the corrections are small as we will show in Appendix C.

In order to give the proofs of Theorems 2.1 and 2.2, let us summarize the results of the estimates for $\mathbf{E}_\omega[\mathcal{M}_{s,\mathrm{in}}], \mathbf{E}_\omega[\mathcal{M}_{s,\mathrm{out}}]$ and $\mathbf{E}_\omega[\mathcal{M}_{s,\mathrm{edge}}]$. For the details of the calculations, see the corresponding Appendices.

5.2.1 Non-interacting case

Consider first the non-interacting case $U^{(2)} = 0$. We obtain the following estimates:

$$|\mathbf{E}_\omega[\mathcal{M}_{s,\mathrm{in}}]| \le \mathcal{C}_{\mathrm{in},0}\frac{L_y}{\ell_B}\left(\frac{\ell_B}{\delta}\right)^4 \qquad (5.38)$$

from (C.41) in Appendix C.1,

$$|\mathbf{E}_\omega[\mathcal{M}_{s,\mathrm{out}}]| \le \mathcal{C}_{\mathrm{out},0}^{(1)}\frac{L_y}{\ell_B}\left(\frac{\ell_B}{\delta}\right)^4 + \mathcal{C}_{\mathrm{out},0}^{(2)}\left(\frac{\ell_B}{\delta}\right)^2 \qquad (5.39)$$

from (D.13) in Appendix D.1, and

$$|\mathbf{E}_\omega[\mathcal{M}_{s,\mathrm{edge}}]| \le \mathcal{C}_{\mathrm{edge},0}\frac{\delta}{L_y} \qquad (5.40)$$

from (E.3) in Appendix E. Here $\mathcal{C}_{\text{in},0}, \mathcal{C}^{(1)}_{\text{out},0}, \mathcal{C}^{(2)}_{\text{out},0}$ and $\mathcal{C}_{\text{edge},0}$ are positive constants which are independent of L_x, L_y. By choosing

$$\delta = \ell_B \left(\frac{L_y}{\ell_B}\right)^{2/5}, \tag{5.41}$$

we get

$$|\mathbf{E}_\omega[\mathcal{M}_s]| \le |\mathbf{E}_\omega[\mathcal{M}_{s,\text{in}}]| + |\mathbf{E}_\omega[\mathcal{M}_{s,\text{out}}]| + |\mathbf{E}_\omega[\mathcal{M}_{s,\text{edge}}]| \le \mathcal{C}_0 \left(\frac{\ell_B}{L_y}\right)^{3/5} \tag{5.42}$$

with a positive constant $\mathcal{C}_0$. Combining this bound, (5.17) and (5.18), we obtain

$$\left|\sigma_{xy} + \frac{e^2}{h}\nu\right| \le \mathcal{C}_{\text{con},0} \left(\frac{\ell_B}{L_y}\right)^{3/5}, \quad |\sigma_{yy}| \le \mathcal{C}_{\text{con},0} \left(\frac{\ell_B}{L_y}\right)^{3/5}, \tag{5.43}$$

where $\mathcal{C}_{\text{con},0}$ is a positive constant.

5.2.2 Interacting case

Next consider the interacting case $U^{(2)} \neq 0$. We take large L_x, L_y so that $N \ge N_{\min}$, and assume that the single-body potential V_ω is two times continuously differentiable on $\mathbf{R}^2$, i.e., $V_\omega \in C^2(\mathbf{R}^2)$, and satisfies the bound (2.31). Here $N_{\min}$ is a positive number given by (F.16) in Appendix F.2. Then we obtain the following estimates:

$$|\mathbf{E}_\omega[\mathcal{M}_{s,\text{in}}]| \le \mathcal{C}_{\text{in}} \left(\frac{L_x}{\ell_B}\right)^{5/6} \left(\frac{L_y}{\ell_B}\right)^{11/6} \left(\frac{\ell_B}{\delta}\right)^{3} \tag{5.44}$$

from Proposition C.8 in Appendix C.2,

$$|\mathbf{E}_\omega[\mathcal{M}_{s,\text{out}}]| \le \mathcal{C}_{\text{out}} \left(\frac{L_x}{\ell_B}\right)^{5/6} \left(\frac{L_y}{\ell_B}\right)^{11/6} \left(\frac{\ell_B}{\delta}\right)^{3} \tag{5.45}$$

from Proposition D.1 in Appendix D.2, and

$$|\mathbf{E}_\omega[\mathcal{M}_{s,\text{edge}}]| \le \mathcal{C}_{\text{edge}} \frac{\delta}{L_y} \tag{5.46}$$

from (E.3) in Appendix E. Here $\mathcal{C}_{\text{in}}, \mathcal{C}_{\text{out}}$ and $\mathcal{C}_{\text{edge}}$ are positive constants which are independent of L_x, L_y. We choose

$$\delta = \ell_B \left(\frac{L_x}{\ell_B}\right)^{5/24} \left(\frac{L_y}{\ell_B}\right)^{17/24}. \tag{5.47}$$

Then we get

$$|\mathbf{E}_\omega[\mathcal{M}_s]| \le |\mathbf{E}_\omega[\mathcal{M}_{s,\text{in}}]| + |\mathbf{E}_\omega[\mathcal{M}_{s,\text{out}}]| + |\mathbf{E}_\omega[\mathcal{M}_{s,\text{edge}}]| \le \mathcal{C} \left(\frac{L_x}{\ell_B}\right)^{5/24} \left(\frac{\ell_B}{L_y}\right)^{7/24} \tag{5.48}$$

with a positive constant $\mathcal{C}$. In particular, we have

$$|\mathbf{E}_\omega [\mathcal{M}_s]| \le \mathcal{C} \left(\frac{\ell_B}{L}\right)^{1/12} \quad \text{and} \quad \delta = \ell_B \left(\frac{L}{\ell_B}\right)^{11/12} \quad \text{for } L_x = L_y = L. \tag{5.49}$$

Combining these bounds, (5.17) and (5.18), we obtain

$$\left|\sigma_{xy} + \frac{e^2}{h}\nu\right| \le \mathcal{C}_{\rm con} \left(\frac{L_x}{\ell_B}\right)^{5/24} \left(\frac{\ell_B}{L_y}\right)^{7/24}, \quad |\sigma_{yy}| \le \mathcal{C}_{\rm con} \left(\frac{L_x}{\ell_B}\right)^{5/24} \left(\frac{\ell_B}{L_y}\right)^{7/24}, \tag{5.50}$$

and

$$\left|\sigma_{xy} + \frac{e^2}{h}\nu\right| \le \mathcal{C}_{\rm con} \left(\frac{\ell_B}{L}\right)^{1/12}, \quad |\sigma_{yy}| \le \mathcal{C}_{\rm con} \left(\frac{\ell_B}{L}\right)^{1/12} \quad \text{for } L_x = L_y = L. \tag{5.51}$$

Here $\mathcal{C}_{\rm con}$ is a positive constant.

A The Rayleigh-Schrödinger perturbation theories

In this appendix, we apply the Rayleigh-Schrödinger perturbation theories to the non-degenerate and degenerate "ground states" of the present quantum Hall Hamiltonian $H_\omega^{(N)}$ of (5.1). Since there is an excitation gap above the "ground state(s)", this perturbative treatment is justified mathematically in the sense of an asymptotic expansion with respect to a sufficiently weak electric field.[13]

Recall the Hamiltonian

$$H_\omega^{(N)} = H_{\omega,0}^{(N)} + \lambda \tilde{H}^{(N)}, \tag{A.1}$$

where λ is a sufficiently small real parameter. The Schrödinger equation is

$$H_\omega^{(N)} \tilde{\Phi}_\omega^{(N)} = \tilde{E}_\omega^{(N)} \tilde{\Phi}_\omega^{(N)} \tag{A.2}$$

with an energy eigenvalue $\tilde{E}_\omega^{(N)}$. In order to obtain a ground state eigenvector $\tilde{\Phi}_\omega^{(N)}$ and the eigenvalue $\tilde{E}_\omega^{(N)}$ in powers of λ, we treat the Hamiltonian $\tilde{H}^{(N)}$ in (A.1) as a perturbation.

A.1 Non-degenerate case

Consider first the case when the ground state $\Phi_{\omega,0}^{(N)}$ of the unperturbed Hamiltonian $H_{\omega,0}^{(N)}$ is non-degenerate. As usual we expand the eigenvector $\tilde{\Phi}_{\omega,0}^{(N)}$ of the ground state of $H_\omega^{(N)}$ in powers of λ as

$$\tilde{\Phi}_{\omega,0}^{(N)} = \Phi_{\omega,0}^{(N)} + \lambda \sum_{\ell \ne 0} a_\ell \Phi_{\omega,\ell}^{(N)} + \cdots \tag{A.3}$$

in terms of the eigenvectors $\Phi_{\omega,\ell}^{(N)}$ of the unperturbed Hamiltonian $H_{\omega,0}^{(N)}$, and expand the corresponding eigenvalue $\tilde{E}_{\omega,0}^{(N)}$ in powers of λ as

$$\tilde{E}_{\omega,0}^{(N)} = E_{\omega,0}^{(N)} + \lambda E_{\omega,0}^{(N,1)} + \cdots. \tag{A.4}$$

[13]See ref. [26] for the mathematically rigorous perturbation theories.

Here $E_{\omega,0}^{(N)}$ is the energy eigenvalue for the ground state eigenvector $\Phi_{\omega,0}^{(N)}$ of $H_{\omega,0}^{(N)}$. Substituting these expansions and (A.1) into the Schrödinger equation (A.2), one has

$$\begin{aligned}
&\left[H_{\omega,0}^{(N)} + \lambda \tilde{H}^{(N)}\right]\left[\Phi_{\omega,0}^{(N)} + \lambda \sum_{\ell \neq 0} a_\ell \Phi_{\omega,\ell}^{(N)} + \cdots\right] \\
= \quad &\left[E_{\omega,0}^{(N)} + \lambda E_{\omega,0}^{(N,1)} + \cdots\right]\left[\Phi_{\omega,0}^{(N)} + \lambda \sum_{\ell \neq 0} a_\ell \Phi_{\omega,\ell}^{(N)} + \cdots\right].
\end{aligned} \tag{A.5}$$

Immediately,

$$H_{\omega,0}^{(N)} \Phi_{\omega,0}^{(N)} = E_{\omega,0}^{(N)} \Phi_{\omega,0}^{(N)}, \tag{A.6}$$

in the zero-th order of λ, and

$$\tilde{H}^{(N)} \Phi_{\omega,0}^{(N)} + \sum_{\ell \neq 0} a_\ell H_{\omega,0}^{(N)} \Phi_{\omega,\ell}^{(N)} = E_{\omega,0}^{(N,1)} \Phi_{\omega,0}^{(N)} + E_{\omega,0}^{(N)} \sum_{\ell \neq 0} a_\ell \Phi_{\omega,\ell}^{(N)} \tag{A.7}$$

in the first order of λ. Taking the inner product with $\Phi_{\omega,\ell}^{(N)}$ ($\ell \neq 0$) in both sides of (A.7), one has

$$\left\langle \Phi_{\omega,\ell}^{(N)}, \tilde{H}^{(N)} \Phi_{\omega,0}^{(N)} \right\rangle + a_\ell E_{\omega,\ell}^{(N)} = E_{\omega,0}^{(N)} a_\ell. \tag{A.8}$$

Here we have taken $\{\Phi_{\omega,\ell}^{(N)}\}$ to be the orthonormal complete system. As a result, the coefficient a_ℓ is

$$a_\ell = \frac{1}{E_{\omega,0}^{(N)} - E_{\omega,\ell}^{(N)}} \left\langle \Phi_{\omega,\ell}^{(N)}, \tilde{H}^{(N)} \Phi_{\omega,0}^{(N)} \right\rangle. \tag{A.9}$$

Substituting this into (A.3), one has

$$\tilde{\Phi}_{\omega,0}^{(N)} = \Phi_{\omega,0}^{(N)} + \lambda \sum_{\ell \neq 0} \Phi_{\omega,\ell}^{(N)} \frac{1}{E_{\omega,0}^{(N)} - E_{\omega,\ell}^{(N)}} \left\langle \Phi_{\omega,\ell}^{(N)}, \tilde{H}^{(N)} \Phi_{\omega,0}^{(N)} \right\rangle + \mathcal{O}(\lambda^2). \tag{A.10}$$

A.2 Degenerate case

In order to treat the "degenerate ground state", we first rewrite the Hamiltonian $H_\omega^{(N)}$ of (A.1) as

$$H_\omega^{(N)} = H_{\omega,0}^{(N)} + \lambda G_\omega^{(N)} \tilde{H}^{(N)} G_\omega^{(N)} + \lambda \tilde{H}_{\mathrm{G}}^{(N)} \tag{A.11}$$

with

$$\tilde{H}_{\mathrm{G}}^{(N)} := G_\omega^{(N)} \tilde{H}^{(N)} (1 - G_\omega^{(N)}) + (1 - G_\omega^{(N)}) \tilde{H}^{(N)} G_\omega^{(N)} + (1 - G_\omega^{(N)}) \tilde{H}^{(N)} (1 - G_\omega^{(N)}), \tag{A.12}$$

where $G_\omega^{(N)}$ is the orthogonal projection onto the sector spanned by the "ground state" eigenvectors $\Phi_{\omega,(0,\mu)}^{(N)}$ of $H_{\omega,0}^{(N)}$. In the present case, we formally treat the Hamiltonian $\tilde{H}_{\mathrm{G}}^{(N)}$ as a perturbation, although the second term in the right-hand side of (A.11) is still a small perturbation. Let $\Phi_{\omega,(0,\mu)}^{(N,0)}$ be the q eigenvectors of the "unperturbed" Hamiltonian $H_{\omega,0}^{(N)} + \lambda G_\omega^{(N)} \tilde{H}^{(N)} G_\omega^{(N)}$, and let $E_{\omega,(0,\mu)}^{(N,0)}$ be the corresponding energy eigenvalues. We take $\left\{\Phi_{\omega,(0,\mu)}^{(N,0)}\right\}$ to be an orthonormal system. Clearly

$$E_{\omega,(0,\mu)}^{(N,0)} = E_{\omega,(0,\mu)}^{(N)} + \mathcal{O}(\lambda), \tag{A.13}$$

where $E^{(N)}_{\omega,(0,\mu)}$ are the "ground state" energy eigenvalues of the Hamiltonian $H^{(N)}_{\omega,0}$. In the same way as in the preceding Section A.1, the "ground state" eigenvector $\tilde{\Phi}^{(N)}_{\omega,(0,\mu)}$ of $H^{(N)}_{\omega}$ is expanded as

$$\tilde{\Phi}^{(N)}_{\omega,(0,\mu)} = \Phi^{(N,0)}_{\omega,(0,\mu)} + \lambda \sum_{\ell\neq 0} a_\ell \Phi^{(N)}_{\omega,\ell} + \cdots \tag{A.14}$$

and expand the corresponding energy eigenvalue $\tilde{E}^{(N)}_{\omega,(0,\mu)}$ as

$$\tilde{E}^{(N)}_{\omega,(0,\mu)} = E^{(N,0)}_{\omega,(0,\mu)} + \lambda E^{(N,0)}_{\omega,(0,\mu)} + \cdots. \tag{A.15}$$

Substituting these into the Schrödinger equation, one has

$$\tilde{H}^{(N)}_{\mathrm{G}} \Phi^{(N,0)}_{\omega,(0,\mu)} + H^{(N)}_{\omega,0} \sum_{\ell\neq 0} a_\ell \Phi^{(N)}_{\omega,\ell} = E^{(N)}_{\omega,(0,\mu)} \sum_{\ell\neq 0} a_\ell \Phi^{(N)}_{\omega,\ell} + E^{(N,1)}_{\omega,0} \Phi^{(N)}_{\omega,(0,\mu)}, \tag{A.16}$$

where we have used (A.13). Taking the inner product with $\Phi^{(N)}_{\omega,\ell}$ with $\ell \neq 0$ in both sides, one gets

$$a_\ell = \frac{1}{E^{(N)}_{\omega,(0,\mu)} - E^{(N)}_{\omega,\ell}} \left\langle \Phi^{(N)}_{\omega,\ell}, \tilde{H}^{(N)} \Phi^{(N,0)}_{\omega,(0,\mu)} \right\rangle. \tag{A.17}$$

Substituting this into (A.14), one has

$$\tilde{\Phi}^{(N)}_{\omega,(0,\mu)} = \Phi^{(N,0)}_{\omega,(0,\mu)} + \lambda \sum_{\ell\neq 0} \Phi^{(N)}_{\omega,\ell} \frac{1}{E^{(N)}_{\omega,(0,\mu)} - E^{(N)}_{\omega,\ell}} \left\langle \Phi^{(N)}_{\omega,\ell}, \tilde{H}^{(N)} \Phi^{(N,0)}_{\omega,(0,\mu)} \right\rangle + \cdots. \tag{A.18}$$

B Matrix elements of the quantum Hall systems with disorder

In this appendix, we study the properties of some matrix elements (5.16) appeared in the expressions (5.17), (5.18) of the conductivities.

B.1 The single electron Landau Hamiltonian with disorder

Consider first the single electron Landau Hamiltonian

$$\mathcal{H}_\omega = \frac{1}{2m_e} \left[(p_x - eBy)^2 + p_y^2 \right] + V_\omega(x,y) \tag{B.1}$$

with the periodic boundary conditions (3.9). The single-electron potential V_ω with disorder satisfies the periodic boundary conditions (2.4) and the condition (2.5) of boundedness.

Lemma B.1 *Let φ_ω be an eigenvector of the Hamiltonian $\mathcal{H}_\omega$ of (B.1). Then the translate $t^{(y)}(y_k)\varphi_\omega$ is an eigenvector of the Hamiltonian $\mathcal{H}_{\omega'}$ with the potential $V_{\omega'}$ given by*

$$V_{\omega'}(x,y) = V_\omega(x, y - y_k). \tag{B.2}$$

Here $y_k = \hbar k/(eB)$ with $k = 2\pi n/L_x$, $(n \in \mathbf{Z})$.

Proof: From the assumption $\mathcal{H}_\omega \varphi_\omega = \mathcal{E}_\omega \varphi_\omega$, we have

$$\mathcal{E}_\omega t^{(y)}(y_k)\varphi_\omega = t^{(y)}(y_k)\mathcal{H}_\omega \varphi_\omega = t^{(y)}(y_k)\mathcal{H}_\omega \left[t^{(y)}(y_k)\right]^{-1} t^{(y)}(y_k)\varphi_\omega = \mathcal{H}_{\omega'} t^{(y)}(y_k)\varphi_\omega. \tag{B.3}$$

■

In the same way, we have

Lemma B.2 *Let $\varphi_{\omega,\ell}$ be an eigenvector of the Hamiltonian $\mathcal{H}_\omega$ of (B.1) with the eigenvalue $\mathcal{E}_{\omega,\ell}$. Let $\varphi_{\omega',\ell} = t^{(y)}(2y_0)R\varphi_{\omega,\ell}$, where R is the reflection operator defined in (3.18). Then $\varphi_{\omega',\ell}$ is an eigenvector of the Hamiltonian $\mathcal{H}_{\omega'}$ with the random potential $V_{\omega'}$ given by*

$$V_{\omega'}(x, y) = V_\omega(-x, 2y_0 - y). \tag{B.4}$$

Here $y_0 = \hbar k_0/(eB)$ with $k_0 = 2\pi n_0/L_x$, $(n_0 \in \mathbf{Z})$. The corresponding eigenvalue $\mathcal{E}_{\omega',\ell}$ is equal to $\mathcal{E}_{\omega,\ell}$. Further the system $\{\varphi_{\omega',\ell}\}$ is an orthogonal complete system if the original system $\{\varphi_{\omega,\ell}\}$ of the eigenvectors is an orthogonal complete system.

Since π_s is invariant under the magnetic translations $t^{(x)}(\cdots)$ and $t^{(y)}(\cdots)$, one can easily obtain the following lemma:

Lemma B.3 *Let V_ω be a random potential, and let $V_{\omega'}$ be the random potential given by (B.4). Let $\varphi_{\omega,\ell}$ be the eigenvectors of the Hamiltonian $\mathcal{H}_\omega$, and let $\varphi_{\omega',\ell} = t^{(y)}(2y_0)R\varphi_{\omega,\ell}$. Then the following relation is valid:*

$$(\varphi_{\omega,\ell}, \pi_s \varphi_{\omega,\ell'}) = -(\varphi_{\omega',\ell}, \pi_s \varphi_{\omega',\ell'}). \tag{B.5}$$

Let $\varphi_{\omega,\ell}$ be an eigenvector of the Hamiltonian $\mathcal{H}_\omega$. We expand $\varphi_{\omega,\ell}$ in Fourier series as

$$\varphi_{\omega,\ell}(x, y) = L_x^{-1/2} \sum_k e^{ikx} \hat{\varphi}_{\omega,\ell}(k, y). \tag{B.6}$$

Since the vector $\varphi_{\omega,\ell}$ satisfies the periodic boundary condition $\varphi_{\omega,\ell}(x, y) = t_y(L_y)\varphi_{\omega,\ell}(x, y)$, we have

$$\hat{\varphi}_{\omega,\ell}(k, y) = \hat{\varphi}_{\omega,\ell}(k - K, y - L_y) \tag{B.7}$$

as in (3.27) in the proof of Lemma 3.2. We define a projection operator as

$$\tilde{P}(k) := \sum_{\ell \in \mathbf{Z}} P(k + \ell K), \tag{B.8}$$

where $P(k)$ is given in (3.34).

Lemma B.4 *Let V_ω be a random potential, and let $V_{\omega'}$ be the translate given by*

$$V_{\omega'}(x, y) = V_\omega(x, y - y_0), \tag{B.9}$$

where $eBy_0 = \hbar k_0 = 2\pi\hbar n_0/L_x$ with an integer n_0. Then

$$\left(\varphi_{\omega',m}, \tilde{P}(k)\pi_s \varphi_{\omega',n}\right) = \left(\varphi_{\omega,m}, \tilde{P}(k - k_0)\pi_s \varphi_{\omega,n}\right). \tag{B.10}$$

Here $\varphi_{\omega,n}$ is an eigenvector of the Hamiltonian $\mathcal{H}_\omega$, and $\varphi_{\omega',n} = t^{(y)}(y_0)\varphi_{\omega,n}$ which is the corresponding eigenvector of $\mathcal{H}_{\omega'}$ as we showed in Lemma B.1.

Proof: Since the vector $\tilde{P}(k)\pi_s\varphi_{\omega',n}$ satisfies the periodic boundary conditions (3.9), one has

$$\begin{aligned}\left(\varphi_{\omega',m}, \tilde{P}(k)\pi_s\varphi_{\omega',n}\right) &= \left(\varphi_{\omega,m}, \left[t^{(y)}(y_0)\right]^{-1}\tilde{P}(k)\pi_s t^{(y)}(y_0)\varphi_{\omega,n}\right) \\ &= \left(\varphi_{\omega,m}, \left[t^{(y)}(y_0)\right]^{-1}\tilde{P}(k)t^{(y)}(y_0)\pi_s\varphi_{\omega,n}\right). \qquad \text{(B.11)}\end{aligned}$$

Therefore it is sufficinet to show

$$\left[t^{(y)}(y_0)\right]^{-1}\tilde{P}(k)t^{(y)}(y_0) = \tilde{P}(k-k_0). \qquad \text{(B.12)}$$

Let f be a function on $\mathbf{R}^2$ such that it has a Fourier expansion

$$f(x,y) = \sum_{k'} e^{ik'x}\hat{f}(k',y). \qquad \text{(B.13)}$$

Then

$$\begin{aligned}\left[t^{(y)}(y_0)\right]^{-1}\tilde{P}(k)t^{(y)}(y_0)f(x,y) &= \left[t^{(y)}(y_0)\right]^{-1}\tilde{P}(k)\sum_{k'} e^{i(k'+k_0)x}\hat{f}(k',y-y_0) \\ &= \left[t^{(y)}(y_0)\right]^{-1}\sum_{\ell} e^{i(k+\ell K)}\hat{f}(k-k_0+\ell K, y-y_0) \\ &= \sum_{\ell} e^{i(k-k_0+\ell K)}\hat{f}(k-k_0+\ell K, y) \\ &= \tilde{P}(k-k_0)f(x,y). \qquad \text{(B.14)}\end{aligned}$$

■

B.2 The N electrons Landau Hamiltonian with disorder

We define the magnetic translation operators for N electrons as

$$T^{(N,x)}(x') := \bigotimes_{j=1}^{N} t_j^{(x)}(x') \qquad \text{(B.15)}$$

and

$$T^{(N,y)}(y') := \bigotimes_{j=1}^{N} t_j^{(y)}(y'). \qquad \text{(B.16)}$$

Further we define the reflection operator for N electrons as

$$R^{(N)} := \bigotimes_{j=1}^{N} R_j. \qquad \text{(B.17)}$$

In the same way as in Section B.1, we have the following two lemmas:

Lemma B.5 *Let $\Phi_\omega^{(N)}$ be an eigenvector of the Hamiltoian $H_{\omega,0}^{(N)}$ of (5.2) with a ramdom potential V_ω, and let $E_\omega^{(N)}$ be the corresponding energy eigenvalue. Let $V_{\omega'}$ be the reflection of the random potentail V_ω with respect to the axes $x=0$ and $y=y_0$, i.e.,*

$$V_{\omega'}(x,y) = V_\omega(-x, 2y_0 - y), \tag{B.18}$$

where $y_0 = \hbar k_0/(eB)$ with $k_0 = 2\pi n_0/L_x$, $(n_0 \in \mathbf{Z})$. Set $\Phi_{\omega'}^{(N)} = T^{(N,y)}(2y_0)R^{(N)}\Phi_\omega^{(N)}$. Then $\Phi_{\omega'}^{(N)}$ is an eigenvector of $H_{\omega',0}^{(N)}$ with the random potential $V_{\omega'}$, and the energy eigenvalue $E_{\omega'}^{(N)}$ is equal to $E_\omega^{(N)}$.

Lemma B.6 *Let V_ω be a ramdom potential, and let $V_{\omega'}$ be the reflection given by*

$$V_{\omega'}(x,y) = V_\omega(-x, 2y_0 - y). \tag{B.19}$$

Here y_0 is the same as in Lemma B.5. Let $\Phi_{\omega,\ell}^{(N)}$ be eigenvectors of the Hamiltonian $H_{\omega,0}^{(N)}$. Then

$$\left\langle \Phi_{\omega',\ell}^{(N)}, \pi_{x,j}\Phi_{\omega',\ell'}^{(N)} \right\rangle = -\left\langle \Phi_{\omega,\ell}^{(N)}, \pi_{x,j}\Phi_{\omega,\ell'}^{(N)} \right\rangle, \tag{B.20}$$

where the vector $\Phi_{\omega',\ell}^{(N)} = T^{(N,y)}(2y_0)R^{(N)}\Phi_\omega^{(N)}$ which are the eigenvectors of the Hamiltonian $H_{\omega',0}^{(N)}$ with the random potential $V_{\omega'}$ as we showed in the preceding Lemma B.5.

Lemma B.7 *Let V_ω be a random potential, and let $V_{\omega'}$ be the translate given by*

$$V_{\omega'}(x,y) = V_\omega(x, y - y_0), \tag{B.21}$$

where $eBy_0 = \hbar k_0 = 2\pi\hbar n_0/L_x$ with an integer n_0. Then the following equalities are valid:

$$\left\langle \Phi_{\omega',\ell}^{(N)}, \pi_{s,j}\Phi_{\omega',\ell'}^{(N)} \right\rangle = \left\langle \Phi_{\omega,\ell}^{(N)}, \pi_{s,j}\Phi_{\omega,\ell'}^{(N)} \right\rangle \tag{B.22}$$

and

$$\left\langle \Phi_{\omega',\ell}^{(N)}, \tilde{P}_j(k)\pi_{s,j}\Phi_{\omega',\ell'}^{(N)} \right\rangle = \left\langle \Phi_{\omega,\ell}^{(N)}, \tilde{P}_j(k-k_0)\pi_{s,j}\Phi_{\omega,\ell'}^{(N)} \right\rangle. \tag{B.23}$$

Here $\Phi_{\omega,n}^{(N)}$ are the eigenvectors of the Hamiltonian $H_{\omega,0}^{(N)}$, and $\Phi_{\omega',n}^{(N)} = T^{(N,y)}(y_0)\Phi_{\omega,n}^{(N)}$ which are the eigenvectors of $H_{\omega',0}^{(N)}$ with the random potential $V_{\omega'}$.

Proof: Since $\pi_{s,j}$ is invariant the magnetic translations, one can easily obtain (B.22). The relation (B.23) follows from the identity (B.12) in Lemma B.4 ■

C Estimate of $\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{in}}\right]$

In this appendix, we estimate the random average of $\mathcal{M}_{s,\mathrm{in}}$ of (5.32), which is given by

$$\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{in}}\right] = \frac{1}{m_e N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{\left[1-G_\omega^{(N)}\right]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}P_{\mathrm{in},i}p_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right]. \tag{C.1}$$

For this purpose, we first want to get the explicit forms of the "corrections from the boundaries" in (5.37).

To begin with, we note the following: Let $\Phi_\omega^{(N)}$ be an N electrons eigenvector of the unperturbed Hamiltonian $H_{\omega,0}^{(N)}$ of (5.2). Clearly this vector can be expanded as

$$\Phi_\omega^{(N)} = \sum_{\{\ell_j\}} a_{\omega,\{\ell_j\}}\mathrm{Asym}\left[\varphi_{\omega,\ell_1}\otimes\varphi_{\omega,\ell_2}\otimes\cdots\otimes\varphi_{\omega,\ell_N}\right] \tag{C.2}$$

in terms of the normalized eigenvectors $\{\varphi_{\omega,\ell}\}$ of the single electron Hamiltonian $\mathcal{H}_\omega$ of (B.1), where Asym$[\cdots]$ stands for the antisymmetrization of a wavefunction, i.e.,

$$\mathrm{Asym}\left[\Phi\right](\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N) := \frac{1}{\sqrt{N!}}\sum_\sigma(-1)^{\ell(\sigma)}\Phi(\mathbf{r}_{\sigma(1)},\mathbf{r}_{\sigma(2)},\ldots,\mathbf{r}_{\sigma(N)}) \tag{C.3}$$

for a function Φ of $(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)$. Here the sum runs over all the permutations σ of $(1,2,\ldots,N)$, and $\ell(\sigma)$ is the number of binary permutations in the permutation σ.

Lemma C.1 *The following relation is valid:*

$$\begin{aligned}
&\mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{1-G_\omega^{(N)}}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}P_{\mathrm{in},i}p_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right]\\
&= 2\mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{1-G_\omega^{(N)}}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}P_{\mathrm{in},i}^{(+)}p_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right],
\end{aligned} \tag{C.4}$$

where

$$P_{\mathrm{in}}^{(+)} = \sum_{k\in\mathcal{F}(I_{\mathrm{in}}^{(+)})}P(k) \tag{C.5}$$

with the interval $I_{\mathrm{in}}^{(+)} = (0, L_y/2-\delta]$.

Proof: Let V_ω be a random potential, and let $V_{\omega'}$ be the reflection of V_ω with respect to the x and y axes, i.e.,

$$V_{\omega'}(x,y) = V_\omega(-x,-y). \tag{C.6}$$

Let $\varphi_{\omega,n}$ and $\varphi_{\omega',n}$ are the normalized eigenvectors of the single electron Hamiltonian $\mathcal{H}_\omega$ of (B.1) with the random potentials V_ω and $V_{\omega'}$, respectively. From Lemma B.2, we can take $\varphi_{\omega',n} = R\varphi_{\omega,n}$. By using the Fourier expansion (B.6) for $\varphi_{\omega,n}$, we have

$$\begin{aligned}
\varphi_{\omega',n}(x,y) &= RL_x^{-1/2}\sum_k e^{ikx}\hat\varphi_{\omega,n}(k,y)\\
&= L_x^{-1/2}\sum_k e^{-ikx}\hat\varphi_{\omega,n}(k,-y)\\
&= L_x^{-1/2}\sum_k e^{ikx}\hat\varphi_{\omega,n}(-k,-y).
\end{aligned} \tag{C.7}$$

This implies

$$\hat{\varphi}_{\omega',n}(k,y) = \hat{\varphi}_{\omega,n}(-k,-y). \tag{C.8}$$

Thereby we have

$$(\varphi_{\omega',m}, P(k)\varphi_{\omega',n}) = (\varphi_{\omega,m}, P(-k)\varphi_{\omega,n}). \tag{C.9}$$

Let $\Phi_{\omega,n}^{(N)}$ be an eigenvector of the Hamiltonian $H_{\omega,0}^{(N)}$ with the random potential V_ω, and let $\Phi_{\omega',n}^{(N)}$ be an eigenvector of the Hamiltonian $H_{\omega',0}^{(N)}$ with the random potential $V_{\omega'}$ of (C.6). From Lemma B.5, we can take $\Phi_{\omega',n}^{(N)} = R^{(N)}\Phi_{\omega,n}^{(N)}$. Combining this with the expansion (C.2) for the vector $\Phi_{\omega,n}^{(N)}$, we have

$$\Phi_{\omega',n}^{(N)} = \sum_{\{\ell_j\}} a_{\omega,\{\ell_j\}}^{(n)} \mathrm{Asym}\left[\varphi_{\omega',\ell_1} \otimes \varphi_{\omega',\ell_2} \otimes \cdots \otimes \varphi_{\omega',\ell_N}\right] \tag{C.10}$$

with $\varphi_{\omega',\ell} = R\varphi_{\omega,\ell}$. Using this expression, one can easily obtain

$$\begin{aligned}&\left\langle \Phi_{\omega',m}^{(N)}, P_i(k)\Phi_{\omega',n}^{(N)} \right\rangle \\ &= \sum_{\{\ell_j\},\{\ell'_j\}} a_{\omega,\{\ell_j\}}^{(m)}{}^{*} a_{\omega,\{\ell'_j\}}^{(n)} \left\langle \mathrm{Asym}\left[\varphi_{\omega',\ell_1} \otimes \cdots \otimes \varphi_{\omega',\ell_N}\right], P_i(k)\mathrm{Asym}\left[\varphi_{\omega',\ell'_1} \otimes \cdots \otimes \varphi_{\omega',\ell'_N}\right] \right\rangle.\end{aligned} \tag{C.11}$$

The matrix elements in the right-hand side are written as

$$\begin{aligned}&\left\langle \mathrm{Asym}\left[\varphi_{\omega',\ell_1} \otimes \varphi_{\omega',\ell_2} \otimes \cdots, \varphi_{\omega',\ell_N}\right], P_i(k)\mathrm{Asym}\left[\varphi_{\omega',\ell'_1} \otimes \varphi_{\omega',\ell'_2} \otimes \cdots \otimes \varphi_{\omega',\ell'_N}\right] \right\rangle \\ &= \frac{1}{N}\begin{cases} \displaystyle\sum_{\ell \in \{\ell_1,\ldots,\ell_N\}} (\varphi_{\omega',\ell}, P(k)\varphi_{\omega',\ell}) & \text{if } \ell_j = \ell'_j \text{ for all } j = 1,2,\ldots,N; \\ \pm\left(\varphi_{\omega',\ell_j}, P(k)\varphi_{\omega',\ell'_m}\right) & \text{if } \{\ell_k\}_{k=1}^N \backslash \{\ell_j\} = \{\ell'_k\}_{k=1}^N \backslash \{\ell'_m\} \text{ and } \ell_j \ne \ell'_m; \\ 0, & \text{otherwise.} \end{cases}\end{aligned} \tag{C.12}$$

Combining (C.9), (C.11) and (C.12), we obtain

$$\left\langle \Phi_{\omega',m}^{(N)}, P_i(-k)\Phi_{\omega',n}^{(N)} \right\rangle = \left\langle \Phi_{\omega,m}^{(N)}, P_i(k)\Phi_{\omega,n}^{(N)} \right\rangle. \tag{C.13}$$

From this and (B.20) in Lemma B.6, we obtain the desired result (C.4). ■

Lemma C.2 *Let V_ω be a random potential, and let $V_{\omega'}$ be the reflection given by*

$$V_{\omega'}(x,y) = V_\omega(-x, 2y_k - y) \tag{C.14}$$

with $y_k > 0$. Let $\varphi_{\omega,n}$ be the normalized eigenvectors of the single electron Hamiltonian $\mathcal{H}_\omega$ of (B.1)with the potential V_ω, and let $\varphi_{\omega',n} = t^{(y)}(2y_k)R\varphi_{\omega,n}$ which are the eigenvectors of $\mathcal{H}_{\omega'}$ with $V_{\omega'}$ from Lemma B.2. Then the following relation is valid:

$$(\varphi_{\omega',\ell}, P(k)\varphi_{\omega',\ell'}) = (\varphi_{\omega,\ell}, P(k)\varphi_{\omega,\ell'}) + (\varphi_{\omega,\ell}, \tilde{\chi}_k P(k-K)\varphi_{\omega,\ell'}) - (\varphi_{\omega,\ell}, \tilde{\chi}_k P(k)\varphi_{\omega,\ell'}), \tag{C.15}$$

where $K = L_y/\ell_B^2$*, and*

$$\tilde{\chi}_k(y) := \begin{cases} 1 & \text{if } \ y \in [-L_y/2, -L_y/2 + 2y_k]; \\ 0, & \text{otherwise.} \end{cases} \tag{C.16}$$

Proof: By using the Fourier expansion (B.6) for $\varphi_{\omega,n}$, we have

$$\begin{aligned} \varphi_{\omega',n}(x,y) &= t^{(y)}(2y_k)L_x^{-1/2}\sum_{k'} e^{-ik'x}\hat{\varphi}_{\omega,n}(k',-y) \\ &= L_x^{-1/2}\sum_{k'} e^{i(2k-k')x}\hat{\varphi}_{\omega,n}(k', 2y_k - y) \\ &= L_x^{-1/2}\sum_{k''} e^{ik''x}\hat{\varphi}_{\omega,n}(2k-k'', 2y_k - y). \end{aligned} \tag{C.17}$$

Thereby we get

$$(\varphi_{\omega',\ell}, P(k)\varphi_{\omega',\ell'}) = \int_{-L_y/2}^{L_y/2} dy \ \ [\hat{\varphi}_{\omega,\ell}(k, 2y_k - y)]^* \, \hat{\varphi}_{\omega,\ell'}(k, 2y_k - y). \tag{C.18}$$

Further we can rewrite the right-hand side as

$$\begin{aligned} & (\varphi_{\omega',\ell}, P(k)\varphi_{\omega',\ell'}) \\ = \ & \int_{-L_y/2+2y_k}^{L_y/2+2y_k} d\tilde{y} \ \ [\hat{\varphi}_{\omega,\ell}(k,\tilde{y})]^* \, \hat{\varphi}_{\omega,\ell'}(k,\tilde{y}) \\ = \ & (\varphi_{\omega,\ell}, P(k)\varphi_{\omega,\ell'}) \\ + \ & \int_{L_y/2}^{L_y/2+2y_k} d\tilde{y} \ \ [\hat{\varphi}_{\omega,\ell}(k,\tilde{y})]^* \, \hat{\varphi}_{\omega,\ell'}(k,\tilde{y}) - \int_{-L_y/2}^{-L_y/2+2y_k} d\tilde{y} \ \ [\hat{\varphi}_{\omega,\ell}(k,\tilde{y})]^* \, \hat{\varphi}_{\omega,\ell'}(k,\tilde{y}) \\ = \ & (\varphi_{\omega,\ell}, P(k)\varphi_{\omega,\ell'}) + \int_{-L_y/2}^{-L_y/2+2y_k} d\tilde{y} \ \ [\hat{\varphi}_{\omega,\ell}(k-K,\tilde{y})]^* \, \hat{\varphi}_{\omega,\ell'}(k-K,\tilde{y}) \\ & - \int_{-L_y/2}^{-L_y/2+2y_k} d\tilde{y} \ \ [\hat{\varphi}_{\omega,\ell}(k,\tilde{y})]^* \, \hat{\varphi}_{\omega,\ell'}(k,\tilde{y}) \\ = \ & (\varphi_{\omega,\ell}, P(k)\varphi_{\omega,\ell'}) + (\varphi_{\omega,\ell}, \tilde{\chi}_k P(k-K)\varphi_{\omega,\ell'}) - (\varphi_{\omega,\ell}, \tilde{\chi}_k P(k)\varphi_{\omega,\ell'})\,. \end{aligned} \tag{C.19}$$

Here we have used (B.7) for getting the third equality. ■

Lemma C.3 *Let* V_ω *be a ramdom potential, and let* $V_{\omega'}$ *be the reflection given by*

$$V_{\omega'}(x,y) = V_\omega(-x, 2y_k - y). \tag{C.20}$$

Here y_k *is the same as in the preceding Lemma C.2. Let* $\Phi_{\omega,n}^{(N)}$ *be the eigenvectors of the Hamiltonian* $H_{\omega,0}^{(N)}$ *with the random potential* V_ω*, and let* $\Phi_{\omega',n}^{(N)} = T^{(N,y)}(2y_k)R^{(N)}\Phi_{\omega,n}^{(N)}$ *which are the eigenvectors of the Hamiltonian* $H_{\omega',0}^{(N)}$ *with the random potential* $V_{\omega'}$ *of (C.20), as we showed in Lemma B.5. Then the following relation is valid:*

$$\begin{aligned} & \left\langle \Phi_{\omega',n}^{(N)}, P_i(k)\Phi_{\omega',0}^{(N)} \right\rangle \\ = \ & \left\langle \Phi_{\omega,n}^{(N)}, P_i(k)\Phi_{\omega,0}^{(N)} \right\rangle + \left\langle \Phi_{\omega,n}^{(N)}, \tilde{\chi}_{k,i} P_i(k-K)\Phi_{\omega,0}^{(N)} \right\rangle - \left\langle \Phi_{\omega,n}^{(N)}, \tilde{\chi}_{k,i} P_i(k)\Phi_{\omega,0}^{(N)} \right\rangle. \end{aligned} \tag{C.21}$$

Proof: In the same way as in the proof of Lemma C.1, we have the expressions (C.11) and (C.12) also for the random potential $V_{\omega'}$ of (C.20). Combining these with (C.15), we obtain the desired result (C.21). ■

Using the above result (C.21) and Lemma B.6, we have

$$\begin{aligned}
& \left\langle \Phi_{\omega',0}^{(N)}, \pi_{s,j}\Phi_{\omega',\ell}^{(N)}\right\rangle \frac{1}{E_{\omega',0}^{(N)}-E_{\omega',\ell}^{(N)}}\left\langle \Phi_{\omega',\ell}^{(N)}, P_{\mathrm{in},i}^{(+)}p_{x,i}\Phi_{\omega',0}^{(N)}\right\rangle \\
= & \sum_{k\in\mathcal{F}(I_{\mathrm{in}}^{(+)})} \hbar k \left\langle \Phi_{\omega',0}^{(N)}, \pi_{s,j}\Phi_{\omega',\ell}^{(N)}\right\rangle \frac{1}{E_{\omega',0}^{(N)}-E_{\omega',\ell}^{(N)}}\left\langle \Phi_{\omega',\ell}^{(N)}, P_i(k)\Phi_{\omega',0}^{(N)}\right\rangle \\
= & -\sum_{k\in\mathcal{F}(I_{\mathrm{in}}^{(+)})} \hbar k \left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\Phi_{\omega,\ell}^{(N)}\right\rangle \frac{1}{E_{\omega,0}^{(N)}-E_{\omega,\ell}^{(N)}} \\
\times & \left[\left\langle \Phi_{\omega,\ell}^{(N)}, P_i(k)\Phi_{\omega,0}^{(N)}\right\rangle + \left\langle \Phi_{\omega,\ell}^{(N)}, \tilde{\chi}_{k,i}P_i(k-K)\Phi_{\omega,0}^{(N)}\right\rangle - \left\langle \Phi_{\omega,\ell}^{(N)}, \tilde{\chi}_{k,i}P_i(k)\Phi_{\omega,0}^{(N)}\right\rangle\right]
\end{aligned} \tag{C.22}$$

for $\ell \neq 0$. Taking the random average in both sides, we get

$$\begin{aligned}
& 2\mathbf{E}_\omega\left[\sum_{k\in\mathcal{F}(I_{\mathrm{in}}^{(+)})} \hbar k \left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\Phi_{\omega,\ell}^{(N)}\right\rangle \frac{1}{E_{\omega,0}^{(N)}-E_{\omega,\ell}^{(N)}}\left\langle \Phi_{\omega,\ell}^{(N)}, P_i(k)\Phi_{\omega,0}^{(N)}\right\rangle\right] \\
= & \mathbf{E}_\omega\left[\sum_{k\in\mathcal{F}(I_{\mathrm{in}}^{(+)})} \hbar k \left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\Phi_{\omega,\ell}^{(N)}\right\rangle \frac{1}{E_{\omega,0}^{(N)}-E_{\omega,\ell}^{(N)}}\left\langle \Phi_{\omega,\ell}^{(N)}, \tilde{\chi}_{k,i}P_i(k-K)\Phi_{\omega,0}^{(N)}\right\rangle\right] \\
- & \mathbf{E}_\omega\left[\sum_{k\in\mathcal{F}(I_{\mathrm{in}}^{(+)})} \hbar k \left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\Phi_{\omega,\ell}^{(N)}\right\rangle \frac{1}{E_{\omega,0}^{(N)}-E_{\omega,\ell}^{(N)}}\left\langle \Phi_{\omega,\ell}^{(N)}, \tilde{\chi}_{k,i}P_i(k)\Phi_{\omega,0}^{(N)}\right\rangle\right]
\end{aligned} \tag{C.23}$$

for $\ell \neq 0$. From (C.1), (C.4) and (C.23), we have

$$\begin{aligned}
\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{in}}\right] &= \frac{1}{m_e N}\sum_{i=1}^N\sum_{j=1}^N \mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}P_{\mathrm{in},i}p_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right] \\
&= \frac{1}{m_e N}\sum_{i=1}^N\sum_{j=1}^N \mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}\tilde{Q}_{\mathrm{in},i}^{(+)}(p_{x,i}+\hbar K)\Phi_{\omega,0}^{(N)}\right\rangle\right] \\
&- \frac{1}{m_e N}\sum_{i=1}^N\sum_{j=1}^N \mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}Q_{\mathrm{in},i}^{(+)}p_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right],
\end{aligned} \tag{C.24}$$

where

$$Q_{\mathrm{in}}^{(+)} := \sum_{k\in\mathcal{F}(I_{\mathrm{in}}^{(+)})} P(k)\tilde{\chi}_k, \tag{C.25}$$

and

$$\tilde{Q}_{\mathrm{in}}^{(+)} := \sum_{k\in\mathcal{F}(I_{\mathrm{in}}^{(+)})} P(k-K)\tilde{\chi}_k. \tag{C.26}$$

Now we estimate $\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{in}}\right]$ by using the expression (C.24).

C.1 Non-interacting case

Consider first the non-interacting case, i.e., $U^{(2)}=0$. Then $\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{in}}\right]$ of (C.24) can be written as

$$\begin{aligned}\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{in}}\right] &= \frac{1}{m_eN}\sum_{n\le N}\mathbf{E}_\omega\left[\left(\varphi_{\omega,n},\pi_s\frac{\mathcal{P}_>}{\mathcal{E}_{\omega,n}-\mathcal{H}_\omega}\tilde{Q}_{\mathrm{in}}^{(+)}(p_x+\hbar K)\varphi_{\omega,n}\right)\right]\\ &- \frac{1}{m_eN}\sum_{n\le N}\mathbf{E}_\omega\left[\left(\varphi_{\omega,n},\pi_s\frac{\mathcal{P}_>}{\mathcal{E}_{\omega,n}-\mathcal{H}_\omega}Q_{\mathrm{in}}^{(+)}p_x\varphi_{\omega,n}\right)\right] \qquad \text{(C.27)}\end{aligned}$$

in terms of the eigenvectors $\varphi_{\omega,n}$ of the single electron Hamiltonian $\mathcal{H}_\omega$ of (B.1), with the energy eigenvalues $\mathcal{E}_{\omega,n}$, $n=1,2,\ldots$. Here we have taken order $\mathcal{E}_{\omega,m}\le\mathcal{E}_{\omega,n}$ for $m<n$, and $\mathcal{P}_>$ is the projection onto the subspace spanned by all states above the Fermi level, i.e., all the vectors $\varphi_{\omega,n}$ with $n>N$. Without loss of generality, we can assume $V_\omega\ge 0$. Then we have $\mathcal{E}_{\omega,n}\ge 0$ for all indices n.

Let us estimate the matrix elements in the second sum in the right-hand side of (C.27). Using the Schwarz inequality we have

$$\begin{aligned}&\left|\left(\varphi_{\omega,n},\pi_s\frac{\mathcal{P}_>}{\mathcal{E}_{\omega,n}-\mathcal{H}_\omega}Q_{\mathrm{in}}^{(+)}p_x\varphi_{\omega,n}\right)\right|\\ \le\ &\frac{eBL_y}{2}\sqrt{\left(\psi_{\omega,n}^{(s)},Q_{\mathrm{in}}^{(+)}\psi_{\omega,n}^{(s)}\right)\left(\varphi_{\omega,n},Q_{\mathrm{in}}^{(+)}\varphi_{\omega,n}\right)}, \qquad \text{(C.28)}\end{aligned}$$

where

$$\psi_{\omega,n}^{(s)}:=\frac{\mathcal{P}_>}{\mathcal{E}_{\omega,n}-\mathcal{H}_\omega}\pi_s\varphi_{\omega,n}. \qquad \text{(C.29)}$$

Lemma C.4

$$\left(\varphi_{\omega,n},Q_{\mathrm{in}}^{(+)}\varphi_{\omega,n}\right)\le\mathcal{C}_1\left(\frac{\ell_B}{\delta}\right)^4 \quad \textit{for } n\le N \qquad \text{(C.30)}$$

with the positive constant

$$\mathcal{C}_1:=\left(\frac{2}{\hbar\omega_c}\right)^2\left(\mathcal{E}_{0,>}+\|V_\omega\|\right)\left(\mathcal{E}_{0,>}+\|V_\omega\|+4\hbar\omega_c\right). \qquad \text{(C.31)}$$

Here $\mathcal{E}_{0,>}:=\min_{m>N}\{\mathcal{E}_{\omega,m}\}$.

Proof: Note that

$$\sum_{k\in\mathcal{F}(I_{\mathrm{in}}^{(+)})}\int_{-L_y/2}^{-L_y/2+2y_k}dy\ (y-y_k)^4|\hat{\varphi}_{\omega,n}(k,y)|^2$$

$$
\begin{aligned}
&\geq \sum_{k\in\mathcal{F}(I_{\rm in}^{(+)})} \int_{-L_y/2}^{-L_y/2+2y_k} dy \ \left(\frac{L_y}{2} - y_k\right)^4 |\hat{\varphi}_{\omega,n}(k,y)|^2 \\
&\geq \delta^4 \sum_{k\in\mathcal{F}(I_{\rm in}^{(+)})} \int_{-L_y/2}^{-L_y/2+2y_k} dy \ |\hat{\varphi}_{\omega,n}(k,y)|^2 \\
&= \delta^4 \left(\varphi_{\omega,n}, Q_{\rm in}^{(+)}\varphi_{\omega,n}\right). \qquad \text{(C.32)}
\end{aligned}
$$

Combining this with the bound (F.1) in Appendix F.1, we get (C.30). ■

A similar bound for $\left(\psi_{\omega,n}^{(s)}, Q_{\rm in}^{(+)}\psi_{\omega,n}^{(s)}\right)$ in (C.28) can be obtained as follows: In the same way as in the proof of Lemma C.4, we have

$$
\begin{aligned}
&\delta^4 \left(\psi_{\omega,n}^{(s)}, Q_{\rm in}^{(+)}\psi_{\omega,n}^{(s)}\right) \\
&\leq \sum_{k\in\mathcal{F}(I_{\rm in}^{(+)})} \int_{-L_y/2}^{-L_y/2+2y_k} dy \ (y-y_k)^4 \left|\hat{\psi}_{\omega,n}^{(s)}(k,y)\right|^2 \\
&\leq \ell_B^4 \left(\frac{2}{\hbar\omega_c}\right)^2 \left\{\left(\|\mathcal{H}_\omega\psi_{\omega,n}^{(s)}\| + \|V_\omega\|\|\psi_{\omega,n}^{(s)}\|\right)^2 + 4\hbar\omega_c\left[\left(\psi_{\omega,n}^{(s)}, \mathcal{H}_\omega\psi_{\omega,n}^{(s)}\right) + \|V_\omega\|\|\psi_{\omega,n}^{(s)}\|^2\right]\right\}. \qquad \text{(C.33)}
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{\mathcal{H}_\omega}{\mathcal{H}_\omega - \mathcal{E}_{\omega,n}}\mathcal{P}_> &= \frac{\mathcal{H}_\omega - \mathcal{E}_{\omega,n} + \mathcal{E}_{\omega,n}}{\mathcal{H}_\omega - \mathcal{E}_{\omega,n}}\mathcal{P}_> \\
&= \left(1 + \frac{\mathcal{E}_{\omega,n}}{\mathcal{H}_\omega - \mathcal{E}_{\omega,n}}\right)\mathcal{P}_> \\
&\leq \min_{m>N}\left(1 + \frac{\mathcal{E}_{\omega,n}}{\mathcal{E}_{\omega,m} - \mathcal{E}_{\omega,n}}\right)\mathcal{P}_> \\
&= \frac{\mathcal{E}_{0,>}}{\mathcal{E}_{0,>} - \mathcal{E}_{\omega,n}}\mathcal{P}_> \leq \frac{\mathcal{E}_{0,>}}{\Delta E}\mathcal{P}_> \qquad \text{(C.34)}
\end{aligned}
$$

for the indices $n \leq N$. Here ΔE is the lower bound for the energy gap given in (2.26). Clearly $\Delta E \leq \min_{n\leq N}\{\mathcal{E}_{0,>} - \mathcal{E}_{\omega,n}\}$ which we have used for getting the last inequality in (C.34). Using the bound (C.34), we have

$$
\left\|\mathcal{H}_\omega\psi_{\omega,n}^{(s)}\right\|^2 \leq \left(\frac{\mathcal{E}_{0,>}}{\Delta E}\right)^2 \left(\varphi_{\omega,n}, \pi_s^2\varphi_{\omega,n}\right) \leq 2m_e\mathcal{E}_{\omega,n}\left(\frac{\mathcal{E}_{0,>}}{\Delta E}\right)^2. \qquad \text{(C.35)}
$$

Similarly we obtain

$$
\left(\psi_{\omega,n}^{(s)}, \mathcal{H}_\omega\psi_{\omega,n}^{(s)}\right) \leq \frac{2m_e\mathcal{E}_{\omega,n}\mathcal{E}_{0,>}}{(\Delta E)^2}, \qquad \text{(C.36)}
$$

and

$$
\left\|\psi_{\omega,n}^{(s)}\right\|^2 \leq \frac{2m_e\mathcal{E}_{\omega,n}}{(\Delta E)^2}. \qquad \text{(C.37)}
$$

Substituting these bounds into (C.33), we have

Lemma C.5

$$\left(\psi_{\omega,n}^{(s)}, Q_{\rm in}^{(+)}\psi_{\omega,n}^{(s)}\right) \le \mathcal{C}_1 \frac{2m_e\mathcal{E}_{\omega,n}}{(\Delta E)^2}\left(\frac{\ell_B}{\delta}\right)^4 \quad \text{for } n \le N. \tag{C.38}$$

From the bound (C.28) and Lemmas C.4 and C.5, we have

Lemma C.6

$$\left|\left(\varphi_{\omega,n}, \pi_s \frac{\mathcal{P}_>}{\mathcal{E}_{\omega,n}-\mathcal{H}_\omega} Q_{\rm in}^{(+)} p_x \varphi_{\omega,n}\right)\right| \le \frac{m_e\sqrt{\hbar\omega_c\mathcal{E}_{0,>}}}{\sqrt{2}\Delta E}\mathcal{C}_1\frac{L_y}{\ell_B}\left(\frac{\ell_B}{\delta}\right)^4 \quad \text{for } n \le N. \tag{C.39}$$

In the same way, we have the following lemma:

Lemma C.7

$$\left|\left(\varphi_{\omega,n}, \pi_s \frac{\mathcal{P}_>}{\mathcal{E}_{\omega,n}-\mathcal{H}_\omega} \tilde{Q}_{\rm in}^{(+)} (p_x+\hbar K) \varphi_{\omega,n}\right)\right| \le \frac{m_e\sqrt{\hbar\omega_c\mathcal{E}_{0,>}}}{\sqrt{2}\Delta E}\mathcal{C}_1\frac{L_y}{\ell_B}\left(\frac{\ell_B}{\delta}\right)^4 \quad \text{for } n \le N. \tag{C.40}$$

Combining these Lemmas with (C.27), we obtain

$$\left|\mathbf{E}_\omega\left[\mathcal{M}_{s,\rm in}\right]\right| \le \frac{\sqrt{2\hbar\omega_c\mathcal{E}_{0,>}}}{\Delta E}\mathcal{C}_1\frac{L_y}{\ell_B}\left(\frac{\ell_B}{\delta}\right)^4. \tag{C.41}$$

C.2 Interacting case

Next we estimate $\mathbf{E}_\omega[\mathcal{M}_{s,\rm in}]$ of (C.24) in the interacting case $U^{(2)} \ne 0$. As a result we obtain the following proposition:

Proposition C.8 *Suppose that $V_\omega \in C^2(\mathbf{R}^2)$ and V_ω satisfies the bound (2.31) in Theorem 2.2. Then*

$$\left|\mathbf{E}_\omega\left[\mathcal{M}_{s,\rm in}\right]\right| \le \mathcal{C}_{\rm in}\left(\frac{L_x}{\ell_B}\right)^{5/6}\left(\frac{L_y}{\ell_B}\right)^{11/6}\left(\frac{\ell_B}{\delta}\right)^3 \quad \text{for } N \ge N_{\rm min}, \tag{C.42}$$

where $N_{\rm min}$ is a positive number which is independent of the linear dimensions L_x, L_y of the system, and $\mathcal{C}_{\rm in}$ is a positive constant which is independent of the linear dimensions L_x, L_y of the system.

The number $N_{\rm min}$ is given explicitly in (F.16) in Appendix F.2. In the rest of this appendix, we assume $V_\omega \in C^2(\mathbf{R}^2)$.

Let A be a symmetric operator. Then one formally has

$$\begin{aligned} \left\langle \Phi_{\omega,0}^{(N)}, A[1-G_\omega^{(N)}]A\Phi_{\omega,0}^{(N)}\right\rangle &= \left\langle A\Phi_{\omega,0}^{(N)}, [1-G_\omega^{(N)}]A\Phi_{\omega,0}^{(N)}\right\rangle \\ &\le \left\langle A\Phi_{\omega,0}^{(N)}, \frac{H_{\omega,0}^{(N)}-E_{\omega,0}^{(N)}}{\Delta E}A\Phi_{\omega,0}^{(N)}\right\rangle \\ &= \frac{1}{2\Delta E}\left\langle \Phi_{\omega,0}^{(N)}, \left[A,[H_{\omega,0}^{(N)},A]\right]\Phi_{\omega,0}^{(N)}\right\rangle \end{aligned} \tag{C.43}$$

for the ground state $\Phi_{\omega,0}^{(N)}$ of the Hamiltonian $H_{\omega,0}^{(N)}$ of (5.2). Using the techniques developed in [27, 28] with this bound, we obtain the following lemma:

Lemma C.9 *The following bound is valid:*

$$\left\langle \Phi_{\omega,0}^{(N)}, \sum_{i=1}^{N} \pi_{s,i}[1 - G_\omega^{(N)}] \sum_{j=1}^{N} \pi_{s,j} \Phi_{\omega,0}^{(N)} \right\rangle \le m_e \hbar\omega_c \mathcal{C}_2 N \tag{C.44}$$

with the positive constant

$$\mathcal{C}_2 := \frac{1}{2\Delta E}\left[\hbar\omega_c + \ell_B^2 \left\| \frac{\partial^2}{\partial x^2} V_\omega \right\|\right]. \tag{C.45}$$

Proof: We treat only the case with $s = x$ because the other can be treated in the same way. Note that

$$\begin{aligned}
&\left[\sum_{i=1}^{N} \pi_{x,i}, \left[H_{\omega,0}^{(N)}, \sum_{j=1}^{N} \pi_{x,j}\right]\right] \\
= \;& \sum_{i=1}^{N} \left\{ \left[\pi_{x,i}, \left[\frac{p_{y,i}^2}{2m_e}, \pi_{x,i}\right]\right] + [\pi_{x,i}, [V_\omega(\mathbf{r}_i), \pi_{x,i}]] \right\} + \sum_{i,j} \left[\pi_{x,i}, \left[U^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N), \pi_{x,j}\right]\right] \\
= \;& \frac{N(\hbar eB)^2}{m_e} + \sum_{i=1}^{N} \hbar^2 \frac{\partial^2}{\partial x_i^2} V_\omega(\mathbf{r}_i),
\end{aligned} \tag{C.46}$$

where we have used the identity $\left[U^{(N)}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N), \sum_j \pi_{x,j}\right] = 0$ which is due to the assumption that the potential $U^{(N)}$ is a function of only the relative coordinates $\mathbf{r}_{ij} = (x_i - x_j, y_i - y_j)$. Combining this with (C.43), we have the desired bound

$$\begin{aligned}
&\left\langle \Phi_{\omega,0}^{(N)}, \sum_{i=1}^{N} \pi_{x,i}[1 - G_\omega^{(N)}] \sum_{j=1}^{N} \pi_{x,j} \Phi_{\omega,0}^{(N)} \right\rangle \\
\le \;& \frac{N}{2\Delta E}\left[\frac{(\hbar eB)^2}{m_e} + \hbar^2 \left\langle \Phi_{\omega,0}^{(N)}, \frac{\partial^2}{\partial x_i^2} V_\omega(\mathbf{r}_i) \Phi_{\omega,0}^{(N)} \right\rangle\right] \le \frac{N}{2\Delta E}\left[\frac{(\hbar eB)^2}{m_e} + \hbar^2 \left\| \frac{\partial^2}{\partial x^2} V_\omega \right\|\right].
\end{aligned} \tag{C.47}$$

■

We write

$$\Psi_\omega^{(N,s)} = \frac{[1 - G_\omega^{(N)}]}{E_{\omega,0}^{(N)} - H_{\omega,0}^{(N)}} \sum_{j=1}^{N} \pi_{s,j} \Phi_{\omega,0}^{(N)}. \tag{C.48}$$

Lemma C.10 *The following bound is valid:*

$$\left\langle \Psi_\omega^{(N,s)}, \pi_{x,j}^2 \Psi_\omega^{(N,s)} \right\rangle \le \frac{2m_e^2 \hbar\omega_c}{\Delta E} \mathcal{C}_2 \left(1 + \frac{N\tilde{\mathcal{E}}}{\Delta E}\right), \tag{C.49}$$

where $\tilde{\mathcal{E}}$ is a positive constant which is independent of the linear dimensions L_x, L_y of the system, and the constant $\mathcal{C}_2$ is given by (C.45).

Proof: Using an identity

$$\frac{H_{\omega,0}^{(N)}}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}[1-G_\omega^{(N)}]=\left(-1+\frac{E_{\omega,0}^{(N)}}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}\right)[1-G_\omega^{(N)}], \tag{C.50}$$

we have

$$\begin{aligned}\frac{1}{2m_e}\left\langle\Psi_\omega^{(N,s)},\pi_{x,j}^2\Psi_\omega^{(N,s)}\right\rangle &\le \frac{1}{N}\left\langle\Psi_\omega^{(N,s)},H_{\omega,0}^{(N)}\Psi_\omega^{(N,s)}\right\rangle\\ &\le \frac{1}{N}\left\langle\Phi_{\omega,0}^{(N)},\sum_{i=1}^N\pi_{s,i}\frac{[1-G_\omega^{(N)}]}{H_{\omega,0}^{(N)}-E_{\omega,0}^{(N)}}\sum_{j=1}^N\pi_{s,j}\Phi_{\omega,0}^{(N)}\right\rangle\\ &+ \frac{E_{\omega,0}^{(N)}}{N}\left\langle\Phi_{\omega,0}^{(N)},\sum_{i=1}^N\pi_{s,i}\left[\frac{1-G_\omega^{(N)}}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}\right]^2\sum_{j=1}^N\pi_{s,j}\Phi_{\omega,0}^{(N)}\right\rangle\\ &\le \frac{1}{N\Delta E}\left(1+\frac{E_{\omega,0}^{(N)}}{\Delta E}\right)\left\langle\Phi_{\omega,0}^{(N)},\sum_{i=1}^N\pi_{s,i}\left[1-G_\omega^{(N)}\right]\sum_{j=1}^N\pi_{s,j}\Phi_{\omega,0}^{(N)}\right\rangle.\end{aligned} \tag{C.51}$$

Combining this, the bound (C.44) of Lemma C.9 and Lemma G.1 in Appendix G, we obtain the desired bound (C.49). ■

Proof of Proposition C.8: In terms of the vector $\Psi_\omega^{(N,s)}$ of (C.48), $\mathbf{E}_\omega[\mathcal{M}_{s,\mathrm{in}}]$ can be written as

$$\begin{aligned}\mathbf{E}_\omega[\mathcal{M}_{s,\mathrm{in}}] &= \frac{1}{m_eN}\sum_{i=1}^N\mathbf{E}_\omega\left[\left\langle\Psi_\omega^{(N,s)},\tilde{Q}_{\mathrm{in},i}^{(+)}(p_{x,i}+\hbar K)\Phi_{\omega,0}^{(N)}\right\rangle\right]\\ &- \frac{1}{m_eN}\sum_{i=1}^N\mathbf{E}_\omega\left[\left\langle\Psi_\omega^{(N,s)},Q_{\mathrm{in},i}^{(+)}p_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right].\end{aligned} \tag{C.52}$$

Using the Schwarz inequality, we have

$$\left|\left\langle\Psi_\omega^{(N,s)},Q_{\mathrm{in},i}^{(+)}p_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right|\le\frac{eBL_y}{2}\sqrt{\left\langle\Psi_\omega^{(N,s)},Q_{\mathrm{in},i}^{(+)}\Psi_\omega^{(N,s)}\right\rangle\left\langle\Phi_{\omega,0}^{(N)},Q_{\mathrm{in},i}^{(+)}\Phi_{\omega,0}^{(N)}\right\rangle}. \tag{C.53}$$

In the same way as in the proof of Lemma C.4, we obtain

$$\left\langle\Psi_\omega^{(N,s)},Q_{\mathrm{in},i}^{(+)}\Psi_\omega^{(N,s)}\right\rangle\le\frac{2m_e}{\Delta E}\mathcal{C}_2\left(1+\frac{N\tilde{\mathcal{E}}}{\Delta E}\right)\left(\frac{\ell_B}{\delta}\right)^2, \tag{C.54}$$

and

$$\left\langle\Phi_{\omega,0}^{(N)},Q_{\mathrm{in},i}^{(+)}\Phi_{\omega,0}^{(N)}\right\rangle\le\left(C_3N^{2/3}+\mathcal{C}_4\right)\left(\frac{\ell_B}{\delta}\right)^4, \tag{C.55}$$

where we have used the bound (C.49) and Proposition F.2. Substituting these bounds into (C.53), we get

$$\left|\left\langle\Psi_\omega^{(N,s)},Q_{\mathrm{in},i}^{(+)}p_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right|\le m_e\sqrt{\frac{\hbar\omega_c}{2\Delta E}\left(1+\frac{N\tilde{\mathcal{E}}}{\Delta E}\right)\mathcal{C}_2(C_3+\mathcal{C}_4N^{-2/3})}\times N^{1/3}\left(\frac{L_y}{\ell_B}\right)\left(\frac{\ell_B}{\delta}\right)^3. \tag{C.56}$$

Similarly we have

$$\begin{aligned}
&\left|\left\langle \Psi_\omega^{(N,s)}, \tilde{Q}_{\mathrm{in},i}^{(+)}(p_{x,i}+\hbar K)\Phi_{\omega,0}^{(N)}\right\rangle\right| \\
\le\ & m_e\sqrt{\frac{\hbar\omega_c}{2\Delta E}\left(1+\frac{N\tilde{\mathcal{E}}}{\Delta E}\right)\mathcal{C}_2(C_3+\mathcal{C}_4N^{-2/3})}\times N^{1/3}\left(\frac{L_y}{\ell_B}\right)\left(\frac{\ell_B}{\delta}\right)^3. \qquad \text{(C.57)}
\end{aligned}$$

Using these bounds for (C.52), we obtain

$$|\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{in}}\right]| \le C'_{\mathrm{in}}N^{5/6}\frac{L_y}{\ell_B}\left(\frac{\ell_B}{\delta}\right)^3 \qquad \text{(C.58)}$$

with the constant

$$C'_{\mathrm{in}} = \sqrt{\frac{2\hbar\omega_c}{\Delta E}\left(\frac{1}{N}+\frac{\tilde{\mathcal{E}}}{\Delta E}\right)\mathcal{C}_2(C_3+\mathcal{C}_4N^{-2/3})}. \qquad \text{(C.59)}$$

Consequently we obtain the desired bound (C.42) from $N = \nu M$ with $M = L_xL_yeB/h$. ■

D Estimate of $\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{out}}\right]$

In this appendix we estimate $\mathcal{M}_{s,\mathrm{out}}$ of (5.33). It can be divided into two parts as

$$\mathcal{M}_{s,\mathrm{out}} = \mathcal{M}_{s,\mathrm{out}}^{(1)} + \mathcal{M}_{s,\mathrm{out}}^{(2)} \qquad \text{(D.1)}$$

with

$$\mathcal{M}_{s,\mathrm{out}}^{(1)} = \frac{1}{m_eN}\sum_{i=1}^N\sum_{j=1}^N\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}P_{\mathrm{out},i}\pi_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle \qquad \text{(D.2)}$$

and

$$\mathcal{M}_{s,\mathrm{out}}^{(2)} = \frac{eB}{m_eN}\sum_{i=1}^N\sum_{j=1}^N\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}P_{\mathrm{out},i}y_i\Phi_{\omega,0}^{(N)}\right\rangle. \qquad \text{(D.3)}$$

D.1 Non-interacting case

Consider first the non-interacting case, $U^{(2)} = 0$. Then $\mathcal{M}_{s,\mathrm{out}}^{(1)}$ and $\mathcal{M}_{s,\mathrm{out}}^{(2)}$ can be written as

$$\mathcal{M}_{s,\mathrm{out}}^{(1)} = \frac{1}{m_eN}\sum_{n\le N}\left(\varphi_{\omega,n}, \pi_s\frac{\mathcal{P}_>}{\mathcal{E}_{\omega,n}-\mathcal{H}_\omega}P_{\mathrm{out}}\pi_x\varphi_{\omega,n}\right) = \frac{1}{m_eN}\sum_{n\le N}\left(\psi_{\omega,n}^{(s)}, P_{\mathrm{out}}\pi_x\varphi_{\omega,n}\right), \qquad \text{(D.4)}$$

and

$$\mathcal{M}_{s,\mathrm{out}}^{(2)} = \frac{eB}{m_eN}\sum_{n\le N}\left(\psi_{\omega,n}^{(s)}, P_{\mathrm{out}}y\varphi_{\omega,n}\right) \qquad \text{(D.5)}$$

in terms of the eigenvectors $\varphi_{\omega,n}$ of the single-electron Hamiltonian $\mathcal{H}_\omega$. Here the vector $\psi_{\omega,n}^{(s)}$ is given by (C.29).

Since we have

$$\begin{aligned}&\sum_{k\in K(I_{\rm out})}\int_{-L_y/2}^{L_y/2} dy\ (y-y_k)^4\left|\hat{\varphi}_{\omega,n}(k,y)\right|^2\\ \ge\ &\delta^4\sum_{k\in K(I_{\rm out})}\int_{-L_y/2}^{L_y/2} dy\ \left|\hat{\varphi}_{\omega,n}(k,y)\right|^2=\delta^4\left(\varphi_{\omega,n},P_{\rm out}\varphi_{\omega,n}\right),\end{aligned}\tag{D.6}$$

we obtain

$$(\varphi_{\omega,n},P_{\rm out}\varphi_{\omega,n})\le\mathcal{C}_1\left(\frac{\ell_B}{\delta}\right)^4\quad\text{for } n\le N\tag{D.7}$$

in the same way as in the proof of Lemma C.4. Further we get

$$\left(\psi_{\omega,n}^{(s)},P_{\rm out}\psi_{\omega,n}^{(s)}\right)\le\frac{2m_e\mathcal{E}_{\omega,n}}{(\Delta E)^2}\mathcal{C}_1\left(\frac{\ell_B}{\delta}\right)^4\quad\text{for } n\le N.\tag{D.8}$$

Using the Schwarz inequality and the bound (D.8), we have

$$\frac{1}{m_e}|(\psi_{\omega,n}^{(s)},P_{\rm out}\pi_x\varphi_{\omega,n})|\le\frac{1}{m_e}\sqrt{(\psi_{\omega,n}^{(s)},P_{\rm out}\psi_{\omega,n}^{(s)})}\,\|\pi_x\varphi\|\le\frac{2\mathcal{E}_{\omega,n}}{\Delta E}\sqrt{\mathcal{C}_1}\left(\frac{\ell_B}{\delta}\right)^2.\tag{D.9}$$

Therefore we obtain

$$\left|\mathcal{M}_{s,\rm out}^{(1)}\right|\le\frac{2\mathcal{E}_{0,>}}{\Delta E}\sqrt{\mathcal{C}_1}\left(\frac{\ell_B}{\delta}\right)^2\tag{D.10}$$

for $\mathcal{M}_{s,\rm out}^{(1)}$ of (D.4).

On the other hand we have

$$\begin{aligned}\frac{eB}{m_e}\left|\left(\psi_{\omega,n}^{(s)},P_{\rm out}y\varphi_{\omega,n}\right)\right|\ &\le\ \frac{eBL_y}{2m_e}\sqrt{(\psi_{\omega,n}^{(s)},P_{\rm out}\psi_{\omega,n}^{(s)})(\varphi_{\omega,n}P_{\rm out}\varphi_{\omega,n})}\\ &\le\ \frac{\sqrt{\hbar\omega_c\mathcal{E}_{0,>}}}{\sqrt{2}\Delta E}\mathcal{C}_1\frac{L_y}{\ell_B}\left(\frac{\ell_B}{\delta}\right)^4\end{aligned}\tag{D.11}$$

by using the Schwarz inequality, (D.7) and (D.8). Therefore we obtain

$$\left|\mathcal{M}_{s,\rm out}^{(2)}\right|\le\frac{\sqrt{\hbar\omega_c\mathcal{E}_{0,>}}}{\sqrt{2}\Delta E}\mathcal{C}_1\frac{L_y}{\ell_B}\left(\frac{\ell_B}{\delta}\right)^4\tag{D.12}$$

for $\mathcal{M}_{s,\rm out}^{(2)}$ of (D.5).

Consequently we get

$$|\mathcal{M}_{s,\rm out}|\le\left|\mathcal{M}_{s,\rm out}^{(1)}\right|+\left|\mathcal{M}_{s,\rm out}^{(2)}\right|\le\frac{2\mathcal{E}_{0,>}}{\Delta E}\sqrt{\mathcal{C}_1}\left(\frac{\ell_B}{\delta}\right)^2+\frac{\sqrt{\hbar\omega_c\mathcal{E}_{0,>}}}{\sqrt{2}\Delta E}\mathcal{C}_1\frac{L_y}{\ell_B}\left(\frac{\ell_B}{\delta}\right)^4.\tag{D.13}$$

D.2 Interacting case

In this case we have the following estimate for $\mathbf{E}_\omega[\mathcal{M}_{s,\mathrm{out}}]$:

Proposition D.1 *Suppose that $V_\omega \in C^2(\mathbf{R}^2)$ and V_ω satisfies the bound (2.31) in Theorem 2.2. Then*

$$|\mathbf{E}_\omega[\mathcal{M}_{s,\mathrm{out}}]| \le \mathcal{C}_{\mathrm{out}} \left(\frac{L_x}{\ell_B}\right)^{5/6} \left(\frac{L_y}{\ell_B}\right)^{11/6} \left(\frac{\ell_B}{\delta}\right)^3 \quad \text{for } N \ge N_{\min}, \tag{D.14}$$

where $\mathcal{C}_{\mathrm{out}}$ and $N_{\min}$ are positive constants which are independent of the linear dimensions L_x, L_y of the system. The number $N_{\min}$ is given explicitly by (F.16) in Appendix F.2.

Proof: In terms of the vector $\Psi_\omega^{(N,s)}$ of (C.48), we write $\mathcal{M}_{s,\mathrm{out}}^{(1)}$ of (D.2) as

$$\mathcal{M}_{s,\mathrm{out}}^{(1)} = \frac{1}{m_e N} \sum_{i=1}^N \left\langle \Psi_\omega^{(N,s)}, P_{\mathrm{out},i}\pi_{x,i}\Phi_{\omega,0}^{(N)} \right\rangle. \tag{D.15}$$

Using the Schwarz inequality, we evaluate the matrix element in the right-hand side as

$$\left|\left\langle \Psi_\omega^{(N,s)}, P_{\mathrm{out},i}\pi_{x,i}\Phi_{\omega,0}^{(N)} \right\rangle\right| \le \sqrt{\left\langle \Psi_\omega^{(N,s)}, P_{\mathrm{out},i}\Psi_\omega^{(N,s)} \right\rangle \left\langle \Phi_{\omega,0}^{(N)} P_{\mathrm{out},i}\pi_{x,i}^2\Phi_{\omega,0}^{(N)} \right\rangle}. \tag{D.16}$$

In the same way as in Section C.2, we have

$$\left\langle \Psi_\omega^{(N,s)}, P_{\mathrm{out},i}\Psi_\omega^{(N,s)} \right\rangle \le \frac{2m_e}{\Delta E}\mathcal{C}_2 \left(1 + \frac{N\tilde{\mathcal{E}}}{\Delta E}\right) \left(\frac{\ell_B}{\delta}\right)^2, \tag{D.17}$$

and

$$\left\langle \Phi_{\omega,0}^{(N)} P_{\mathrm{out},i}\pi_{x,i}^2\Phi_{\omega,0}^{(N)} \right\rangle \le \hbar eB \left(\mathcal{C}_3 N^{2/3} + \mathcal{C}_4\right) \left(\frac{\ell_B}{\delta}\right)^2. \tag{D.18}$$

From these three bounds, we estimate $\mathcal{M}_{s,\mathrm{out}}^{(1)}$ of (D.15) as

$$\left|\mathcal{M}_{s,\mathrm{out}}^{(1)}\right| \le \sqrt{\frac{2\hbar\omega_c}{\Delta E}\left(\frac{1}{N} + \frac{\tilde{\mathcal{E}}}{\Delta E}\right)\mathcal{C}_2\left(\mathcal{C}_3 + \mathcal{C}_4 N^{-2/3}\right)} \times N^{5/6} \left(\frac{\ell_B}{\delta}\right)^2. \tag{D.19}$$

Similarly we can write $\mathcal{M}_{s,\mathrm{out}}^{(2)}$ of (D.3) as

$$\mathcal{M}_{s,\mathrm{out}}^{(2)} = \frac{eB}{m_e N} \sum_{i=1}^N \left\langle \Psi_\omega^{(N,s)}, P_{\mathrm{out},i} y_i \Phi_{\omega,0}^{(N)} \right\rangle \tag{D.20}$$

in terms of the vector $\Psi_\omega^{(N,s)}$ of (C.48). Using the Schwarz inequality, we evaluate the matrix element in the right-hand side as

$$\begin{aligned} \left|\left\langle \Psi_\omega^{(N,s)}, P_{\mathrm{out},i} y_i \Phi_{\omega,0}^{(N)} \right\rangle\right| &\le \frac{L_y}{2}\sqrt{\left\langle \Psi_\omega^{(N,s)}, P_{\mathrm{out},i}\Psi_\omega^{(N,s)} \right\rangle \left\langle \Phi_{\omega,0}^{(N)}, P_{\mathrm{out},i}\Phi_{\omega,0}^{(N)} \right\rangle} \\ &\le \frac{m_e}{eB}\sqrt{\frac{\hbar\omega_c}{2\Delta E}\left(\frac{1}{N} + \frac{\tilde{\mathcal{E}}}{\Delta E}\right)\mathcal{C}_2(\mathcal{C}_3 + \mathcal{C}_4 N^{-2/3})} \times N^{5/6}\frac{L_y}{\ell_B}\left(\frac{\ell_B}{\delta}\right)^3, \end{aligned} \tag{D.21}$$

where we have used the bound (D.17) and the bound

$$\left\langle \Phi_{\omega,0}^{(N)}, P_{\mathrm{out},i}\Phi_{\omega,0}^{(N)} \right\rangle \le \left(\mathcal{C}_3 N^{2/3} + \mathcal{C}_4\right)\left(\frac{\ell_B}{\delta}\right)^4. \tag{D.22}$$

The second bound can be derived in the same way as in Section C.2. Substituting (D.21) into (D.20), we get

$$\left|\mathcal{M}_{s,\mathrm{out}}^{(2)}\right| \le \sqrt{\frac{\hbar\omega_c}{2\Delta E}\left(\frac{1}{N} + \frac{\tilde{\mathcal{E}}}{\Delta E}\right)\mathcal{C}_2(\mathcal{C}_3 + \mathcal{C}_4 N^{-2/3})} \times N^{5/6}\frac{L_y}{\ell_B}\left(\frac{\ell_B}{\delta}\right)^3. \tag{D.23}$$

Combining this with (D.19), we obtain

$$\begin{aligned} |\mathcal{M}_{s,\mathrm{out}}| &\le \left|\mathcal{M}_{s,\mathrm{out}}^{(1)}\right| + \left|\mathcal{M}_{s,\mathrm{out}}^{(2)}\right| \\ &\le \sqrt{\frac{2\hbar\omega_c}{\Delta E}\left(\frac{1}{N} + \frac{\tilde{\mathcal{E}}}{\Delta E}\right)\mathcal{C}_2(\mathcal{C}_3 + \mathcal{C}_4 N^{-2/3})} \times \left(1 + \frac{L_y}{2\delta}\right) N^{5/6}\left(\frac{\ell_B}{\delta}\right)^2. \end{aligned} \tag{D.24}$$

Consequently we have the desired bound (D.14) with $N = \nu M$ and $M = L_x L_y eB/h$. ■

E Estimate of $\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{edge}}\right]$

In this appendix, we estimate the random average $\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{edge}}\right]$ of (5.34) which is the contribution near the edges of the system. It can be written as

$$\mathcal{M}_{s,\mathrm{edge}} = \frac{1}{m_e N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell\neq 0}\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\Phi_{\omega,\ell}^{(N)}\right\rangle \frac{1}{E_{\omega,0}^{(N)} - E_{\omega,\ell}^{(N)}}\left\langle \Phi_{\omega,\ell}^{(N)}, P_i(I_{\mathrm{edge}})\pi_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle, \tag{E.1}$$

where

$$I_{\mathrm{edge}} = \bigcup_{\ell\in\mathbf{Z}}(L_y/2 - \delta + \ell L_y, L_y/2 + \delta + \ell L_y). \tag{E.2}$$

As a result, we will obtain

$$|\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{edge}}\right]| \le \mathcal{C}_{\mathrm{edge}}\frac{\delta}{L_y}, \tag{E.3}$$

where $\mathcal{C}_{\mathrm{edge}}$ is a positive constant which is independent of the linear dimensions L_x, L_y of the system in both non-interacting and interacting cases.

From Lemma B.7, we have

$$\begin{aligned} &\mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\Phi_{\omega,\ell}^{(N)}\right\rangle \frac{1}{E_{\omega,0}^{(N)} - E_{\omega,\ell}^{(N)}}\left\langle \Phi_{\omega,\ell}^{(N)}, \tilde{P}_i(k)\pi_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right] \\ &= \frac{1}{M}\mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\Phi_{\omega,\ell}^{(N)}\right\rangle \frac{1}{E_{\omega,0}^{(N)} - E_{\omega,\ell}^{(N)}}\left\langle \Phi_{\omega,\ell}^{(N)}, \pi_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right] \end{aligned} \tag{E.4}$$

for $\ell \neq 0$, where $M = eBL_xL_y/h$. Combining this with (E.1), we obtain

$$|\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{edge}}\right]| \le \frac{2\delta}{m_e L_y N}\left|\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{E}_\omega\left[\left\langle \Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1 - G_\omega^{(N)}]}{E_{\omega,0}^{(N)} - H_{\omega,0}^{(N)}}\pi_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right]\right|. \tag{E.5}$$

E.1 Non-interacting case

Consider first the non-interacting case, $U^{(2)} = 0$. Then (E.5) can be evaluated as

$$\begin{aligned} \left|\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{edge}}\right]\right| &\le \frac{2\delta}{m_e L_y N}\left|\sum_{n\le N}\mathbf{E}_\omega\left[\left(\varphi_{\omega,n}, \pi_s\frac{\mathcal{P}_>}{\mathcal{E}_{\omega,n}-\mathcal{H}_\omega}\pi_x\varphi_{\omega,n}\right)\right]\right| \\ &\le \frac{2\delta}{m_e L_y N}\sum_{n\le N}\mathbf{E}_\omega\left[\frac{1}{\Delta E}\sqrt{\left(\varphi_{\omega,n}, \pi_s^2\varphi_{\omega,n}\right)\left(\varphi_{\omega,n}, \pi_x^2\varphi_{\omega,n}\right)}\right] \le \frac{4\mathcal{E}_{0,>}}{\Delta E}\frac{\delta}{L_y}, \end{aligned} \tag{E.6}$$

where we have used the Schwarz inequality.

E.2 Interacting case

Using the Schwarz inequality and Lemma C.9, we have

$$\begin{aligned} &\left|\sum_{i=1}^N\sum_{j=1}^N\left\langle\Phi_{\omega,0}^{(N)}, \pi_{s,j}\frac{[1-G_\omega^{(N)}]}{E_{\omega,0}^{(N)}-H_{\omega,0}^{(N)}}\pi_{x,i}\Phi_{\omega,0}^{(N)}\right\rangle\right| \\ \le\ &\frac{1}{\Delta E}\sqrt{\left\langle\Phi_{\omega,0}^{(N)}, \sum_{i=1}^N\pi_{s,i}[1-G_\omega^{(N)}]\sum_{j=1}^N\pi_{s,j}\Phi_{\omega,0}^{(N)}\right\rangle\left\langle\Phi_{\omega,0}^{(N)}, \sum_{m=1}^N\pi_{x,m}[1-G_\omega^{(N)}]\sum_{n=1}^N\pi_{x,n}\Phi_{\omega,0}^{(N)}\right\rangle} \\ \le\ &\frac{Nm_e\hbar\omega_c}{\Delta E}\mathcal{C}_2. \end{aligned} \tag{E.7}$$

Substituting this into the right-hand side of (E.5), we obtain

$$\left|\mathbf{E}_\omega\left[\mathcal{M}_{s,\mathrm{edge}}\right]\right| \le \frac{2\hbar\omega_c}{\Delta E}\frac{\delta}{L_y}\mathcal{C}_2. \tag{E.8}$$

F Decay estimate of wavefunctions

In this appendix, we obtain a decay estimate for the Fourier component of a wavefunction for both non-interacting and interacting electrons gases.

F.1 Non-interacting case

The aim of this subsection is to give a proof of the following proposition in the non-interacting case $U^{(2)} = 0$:

Proposition F.1 *Let φ be a wavefunction such that $\|\mathcal{H}_\omega\varphi\| < \infty$. Then*

$$\left|\left(\varphi, [\pi_x/(eB)]^4\varphi\right)\right| \le \ell_B^4\left(\frac{2}{\hbar\omega_c}\right)^2\left\{\left(\|\mathcal{H}_\omega\varphi\| + \|V_\omega\|\|\varphi\|\right)^2 + 4\hbar\omega_c\left[(\varphi, \mathcal{H}_\omega\varphi) + \|V_\omega\|\|\varphi\|^2\right]\right\}. \tag{F.1}$$

In order to see the physical meaning, we write

$$\varphi(x,y)=L_x^{-1/2}\sum_k e^{ikx}\hat{\varphi}(k,y), \tag{F.2}$$

in terms of the Fourier transform. Clearly one has

$$\begin{aligned}&\sum_k\int_{-L_y/2}^{L_y/2}dy(y-y_k)^4|\hat{\varphi}(k,y)|^2\\ \le\ & \ell_B^4\left(\frac{2}{\hbar\omega_c}\right)^2\left\{(\|\mathcal{H}_\omega\varphi\|+\|V_\omega\|\|\varphi\|)^2+4\hbar\omega_c\left[(\varphi,\mathcal{H}_\omega\varphi)+\|V_\omega\|\|\varphi\|^2\right]\right\}\end{aligned} \tag{F.3}$$

with $y_k=\hbar k/(eB)$. This implies that $\hat{\varphi}(k,y)$ decays more rapidly than $|y-y_k|^{-4}$ when $\|\mathcal{H}_\omega\varphi\|<\infty$.

Before giving the proof of Proposition F.1, we shall see a fairly trivial decay estimate for a wavefunction. Let φ be a wavefunction. Then we formally have

$$\frac{1}{2m_e}\left(\varphi,(p_x-eBy)^2\varphi\right)+\frac{1}{2m_e}\left(\varphi,p_y^2\varphi\right)+(\varphi,V_\omega\varphi)=(\varphi,\mathcal{H}_\omega\varphi). \tag{F.4}$$

Clearly we get

$$\frac{1}{2m_e}\left(\varphi,(p_x-eBy)^2\varphi\right)\le(\varphi,\mathcal{H}_\omega\varphi)+\|V_\omega\|\ \|\varphi\|^2, \tag{F.5}$$

and

$$\frac{1}{2m_e}\left(\varphi,p_y^2\varphi\right)\le(\varphi,\mathcal{H}_\omega\varphi)+\|V_\omega\|\ \|\varphi\|^2. \tag{F.6}$$

Combining the first inequality with the Fourier form (F.2), we get

$$\sum_k\int_{-L_y/2}^{L_y/2}dy\ (y-y_k)^2\,|\hat{\varphi}(k,y)|^2\le\frac{2m_e}{e^2B^2}\left[(\varphi,\mathcal{H}_\omega\varphi)+\|V_\omega\|\ \|\varphi\|^2\right], \tag{F.7}$$

where $y_k=\hbar k/(eB)$. This implies that the Fourier component $\hat{\varphi}(k,y)$ decays more rapidly than the inverse square of the distance $|y-y_k|$ when the wavefunction satisfies the condition $|\,(\varphi,\mathcal{H}_\omega\varphi)\,|<\infty$.

In order to obtain the stronger decay bound (F.3), we consider a formal identity,

$$\begin{aligned}\left(\varphi,\mathcal{H}_\omega[y-p_x/(eB)]^2\varphi\right)\ =\ &\frac{e^2B^2}{2m_e}\left(\varphi,[y-p_x/(eB)]^4\varphi\right)+\frac{1}{2m_e}\left(\varphi,p_y^2[y-p_x/(eB)]^2\varphi\right)\\ &+\left(\varphi,V_\omega[y-p_x/(eB)]^2\varphi\right).\end{aligned} \tag{F.8}$$

Since the second term in the right-hand side can be written as

$$\begin{aligned}&\frac{1}{2m_e}\left(\varphi,p_y^2[y-p_x/(eB)]^2\varphi\right)\\ =\ &\frac{1}{2m_e}\left(\varphi,p_y[y-p_x/(eB)]^2p_y\varphi\right)-\frac{i\hbar}{m_e}\left(\varphi,p_y[y-p_x/(eB)]\varphi\right),\end{aligned} \tag{F.9}$$

we have

$$\begin{aligned}
&\frac{e^2B^2}{2m_e}\Big(\varphi,[y-p_x/(eB)]^4\varphi\Big)+\frac{1}{2m_e}\Big(\varphi,p_y[y-p_x/(eB)]^2p_y\varphi\Big)\\
=\;&\Big(\varphi,\mathcal{H}_\omega[y-p_x/(eB)]^2\varphi\Big)+\frac{i\hbar}{m_e}\left(\varphi,p_y[y-p_x/(eB)]\varphi\right)\\
&-\Big(\varphi,V_\omega[y-p_x/(eB)]^2\varphi\Big). \qquad \text{(F.10)}
\end{aligned}$$

In order to get a bound for the first term in the left-hand side, we estimate the right-hand side as follows. The first term in the right-hand side of (F.10) can be evaluated as

$$\left|\Big(\varphi,\mathcal{H}_\omega[y-p_x/(eB)]^2\varphi\Big)\right|\le\|\mathcal{H}_\omega\varphi\|\sqrt{(\varphi,[y-p_x/(eB)]^4\varphi)} \qquad \text{(F.11)}$$

by using the Schwartz inequality. Similarly the second term in the right-hand side of (F.10) can be evaluated as

$$\begin{aligned}
\left|\frac{\hbar}{m_e}\left(\varphi,p_y[y-p_x/(eB)]\varphi\right)\right| &\le \frac{\hbar}{m_e}\|p_y\varphi\|\sqrt{(\varphi,[y-p_x/(eB)]^2\varphi)}\\
&\le \frac{2\hbar}{eB}\Big[(\varphi,\mathcal{H}_\omega\varphi)+\|V_\omega\|\;\|\varphi\|^2\Big], \qquad \text{(F.12)}
\end{aligned}$$

where we have used (F.5), and (F.6). Finally we have

$$\left|\Big(\varphi,V_\omega[y-p_x/(eB)]^2\varphi\Big)\right| \le \|V_\omega\|\;\|\varphi\|\sqrt{(\varphi,[y-p_x/(eB)]^4\varphi)} \qquad \text{(F.13)}$$

for the third term in the right-hand side of (F.10). From these three bounds, we formally obtain

$$\begin{aligned}
\frac{e^2B^2}{2m_e}\Big(\varphi,[y-p_x/(eB)]^4\varphi\Big) &\le \left(\|\mathcal{H}_\omega\varphi\|+\|V_\omega\|\;\|\varphi\|\right)\sqrt{(\varphi,[y-p_x/(eB)]^4\varphi)}\\
&+\frac{2\hbar}{eB}\Big[(\varphi,\mathcal{H}_\omega\varphi)+\|V_\omega\|\;\|\varphi\|^2\Big], \qquad \text{(F.14)}
\end{aligned}$$

where we have used the fact that the second term in the left-hand side of (F.10) is non-negative. From this (F.14), one can easily obtain

$$\begin{aligned}
\sqrt{(\varphi,[y-p_x/(eB)]^4\varphi)} &\le \frac{m_e}{e^2B^2}\left(\|\mathcal{H}_\omega\varphi\|+\|V_\omega\|\|\varphi\|\right)\\
&+\frac{m_e}{e^2B^2}\sqrt{\left(\|\mathcal{H}_\omega\varphi\|+\|V_\omega\|\|\varphi\|\right)^2+4\hbar\omega_c\left[(\varphi,\mathcal{H}_\omega\varphi)+\|V_\omega\|\|\varphi\|^2\right]}\\
&\le \frac{2m_e}{e^2B^2}\sqrt{\left(\|\mathcal{H}_\omega\varphi\|+\|V_\omega\|\|\varphi\|\right)^2+4\hbar\omega_c\left[(\varphi,\mathcal{H}_\omega\varphi)+\|V_\omega\|\|\varphi\|^2\right]}. \qquad \text{(F.15)}
\end{aligned}$$

Thus we have obtained the desired bound (F.1) which is justified for φ satisfying $\|\mathcal{H}_\omega\varphi\|<\infty$.

F.2 Interacting case

Next we consider the interacting case. Our goal of this subsection is to give a proof of Proposition F.2 below which is an extension of the decay bound (F.1) to the interacting electrons gas. We write

$$N_{\min} = \left[\frac{2\nu}{\hbar\omega_c}\left(\tilde{\mathcal{E}} + \|V_\omega\|\right)\right]^{3/2}, \tag{F.16}$$

where $\tilde{\mathcal{E}}$ is an upper bound for the ground state energy per electron $E_{\omega,0}^{(N)}/N$ of the Hamiltonian $H_{\omega,0}^{(N)}$ of (5.2). The constant $\tilde{\mathcal{E}}$ is independent of the linear dimensions L_x, L_y of the system as we show in Lemma G.1 in Appendix G.

Proposition F.2 *Let $\Phi_{\omega,0}^{(N)}$ be the ground state eigenvector of the Hamiltonian $H_{\omega,0}^{(N)}$ with norm one. Then*

$$\left(\frac{1}{eB}\right)^4 \left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 \Phi_{\omega,0}^{(N)} \right\rangle \le \ell_B{}^4 \left(\mathcal{C}_3 N^{2/3} + \mathcal{C}_4\right) \quad \text{for } N \ge N_{\min}, \tag{F.17}$$

where the constants $\mathcal{C}_3$ and $\mathcal{C}_4$ are independent of the linear dimensions L_x, L_y of the system.

Consider an identity

$$\begin{aligned} E_{\omega,0}^{(N)} \left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \Phi_{\omega,0}^{(N)} \right\rangle &= \left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, H_{\omega,0}^{(N)} \Phi_{\omega,0}^{(N)} \right\rangle \\ &= \left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, H_{\omega,0}^{(N,j)} \Phi_{\omega,0}^{(N)} \right\rangle + \left\langle \pi_{x,j} \Phi_{\omega,0}^{(N)}, \left(H_{\omega,0}^{(N)} - H_{\omega,0}^{(N,j)}\right) \pi_{x,j} \Phi_{\omega,0}^{(N)} \right\rangle, \end{aligned} \tag{F.18}$$

where

$$H_{\omega,0}^{(N,j)} := \frac{1}{2m_e}\left(\pi_{x,j}^2 + p_{y,j}^2\right) + V_\omega(\mathbf{r}_j) + U^{(N,j)} \tag{F.19}$$

with

$$U^{(N,j)}(\mathbf{r}_j; \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{j-1}, \mathbf{r}_{j+1}, \ldots, \mathbf{r}_N) = \sum_{\ell \ne j} U^{(2)}(x_j - x_\ell, y_j - y_\ell). \tag{F.20}$$

Clearly the first term in the right-hand side of (F.18) is written as

$$\begin{aligned} \left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, H_{\omega,0}^{(N,j)} \Phi_{\omega,0}^{(N)} \right\rangle &= \frac{1}{2m_e} \left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 \Phi_{\omega,0}^{(N)} \right\rangle + \frac{1}{2m_e} \left\langle \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 p_{y,j}^2 \Phi_{\omega,0}^{(N)} \right\rangle \\ &+ \left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, V_\omega(\mathbf{r}_j) \Phi_{\omega,0}^{(N)} \right\rangle + \left\langle \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 U^{(N,j)} \Phi_{\omega,0}^{(N)} \right\rangle. \end{aligned} \tag{F.21}$$

Note that

$$\pi_{x,j}^2 p_{y,j}^2 = \pi_{x,j} p_{y,j}^2 \pi_{x,j} - \hbar^2 e^2 B^2 - i\hbar e B \left(\pi_{x,j} p_{y,j} + p_{y,j} \pi_{x,j}\right), \tag{F.22}$$

and

$$\pi_{x,j}^2 U^{(N,j)} = \pi_{x,j} U^{(N,j)} \pi_{x,j} - \frac{i\hbar}{2}\left(\pi_{x,j} \frac{\partial}{\partial x_j} U^{(N,j)} + \frac{\partial}{\partial x_j} U^{(N,j)} \pi_{x,j}\right) - \frac{\hbar^2}{2} \frac{\partial^2}{\partial x_j^2} U^{(N,j)}. \tag{F.23}$$

Here we have used the commutation relation $[p_{y,j}, \pi_{x,j}] = i\hbar eB$ for getting the first relation. Since the left-hand side of (F.21) is real from (F.18), we have

$$\begin{aligned}
\left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, H_{\omega,0}^{(N,j)} \Phi_{\omega,0}^{(N)} \right\rangle \;\geq\;& \frac{1}{2m_e} \left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 \Phi_{\omega,0}^{(N)} \right\rangle - \frac{\hbar^2 e^2 B^2}{2m_e} \\
&- \frac{\hbar^2}{2} \left\langle \Phi_{\omega,0}^{(N)}, \frac{\partial^2}{\partial x_j^2} U^{(N,j)} \Phi_{\omega,0}^{(N)} \right\rangle - \|V_\omega\| \sqrt{\left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 \Phi_{\omega,0}^{(N)} \right\rangle}.
\end{aligned} \tag{F.24}$$

Here we have used the Schwarz inequality for evaluating the third term in the right-hand side of (F.21). Substituting this bound (F.24) into (F.18), we obtain

$$\begin{aligned}
& \left(E_{\omega,0}^{(N)} - E_{\omega,0}^{(N-1)} \right) \left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \Phi_{\omega,0}^{(N)} \right\rangle \\
\geq\;& \frac{1}{2m_e} \left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 \Phi_{\omega,0}^{(N)} \right\rangle - \frac{\hbar^2 e^2 B^2}{2m_e} - \frac{\hbar^2}{2} \left\langle \Phi_{\omega,0}^{(N)}, \frac{\partial^2}{\partial x_j^2} U^{(N,j)} \Phi_{\omega,0}^{(N)} \right\rangle \\
-\;& \|V_\omega\| \sqrt{\left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 \Phi_{\omega,0}^{(N)} \right\rangle} \\
\geq\;& \frac{1}{2m_e} \left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 \Phi_{\omega,0}^{(N)} \right\rangle - \frac{\hbar^2 e^2 B^2}{2m_e} - \frac{\hbar^2 \alpha}{N} \left\langle \Phi_{\omega,0}^{(N)}, U^{(N)} \Phi_{\omega,0}^{(N)} \right\rangle \\
-\;& \|V_\omega\| \sqrt{\left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 \Phi_{\omega,0}^{(N)} \right\rangle},
\end{aligned} \tag{F.25}$$

where we have used the assumption (2.11) about $U^{(2)}$ and

$$\left\langle \pi_{x,j} \Phi_{\omega,0}^{(N)}, \left(H_{\omega,0}^{(N)} - H_{\omega,0}^{(N,j)} \right) \pi_{x,j} \Phi_{\omega,0}^{(N)} \right\rangle \geq E_{\omega,0}^{(N-1)} \left\langle \pi_{x,j} \Phi_{\omega,0}^{(N)}, \pi_{x,j} \Phi_{\omega,0}^{(N)} \right\rangle. \tag{F.26}$$

Further the inequality thus obtained is rewritten as

$$\begin{aligned}
& \left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 \Phi_{\omega,0}^{(N)} \right\rangle - 2m_e \|V_\omega\| \sqrt{\left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 \Phi_{\omega,0}^{(N)} \right\rangle} \\
\leq\;& \hbar^2 e^2 B^2 + 2m_e \hbar^2 \alpha \mathcal{U} + \max\left\{ 0, E_{\omega,0}^{(N)} - E_{\omega,0}^{(N-1)} \right\} \times \frac{4m_e^2 E_{\omega,0}^{(N)}}{N} \\
\leq\;& \hbar^2 e^2 B^2 + 2m_e \hbar^2 \alpha \mathcal{U} + \max\left\{ 0, E_{\omega,0}^{(N)} - E_{\omega,0}^{(N-1)} \right\} \times 4m_e^2 \tilde{\mathcal{E}}
\end{aligned} \tag{F.27}$$

by using Lemma G.1 in Appendix G. The energy difference $E_{\omega,0}^{(N)} - E_{\omega,0}^{(N-1)}$ is evaluated as follows:

Lemma F.3 *Let n be an integer such that*

$$n + \frac{1}{2} \geq N_{\min}^{2/3}. \tag{F.28}$$

Then

$$E_{\omega,0}^{(N)} - E_{\omega,0}^{(N-1)} \leq \hbar\omega_c \left(n + \frac{1}{2} \right) + 2\mathcal{C}_5' \frac{N}{\sqrt{2n+1}} + \mathcal{C}_6', \tag{F.29}$$

where $\mathcal{C}_5'$ and $\mathcal{C}_6'$ are positive constant which are independent of the linear dimensions L_x, L_y of the system and of the number N of the electrons.

The proof is given in Appendix H.

Proof of Proposition F.2: From (F.27), we have

$$\begin{aligned}&\left\langle \pi_{x,j}^2 \Phi_{\omega,0}^{(N)}, \pi_{x,j}^2 \Phi_{\omega,0}^{(N)} \right\rangle \\ \le\ & 4m_e^2 \left\| V_\omega \right\|^2 + 4\hbar^2 e^2 B^2 + 8 m_e \hbar^2 \alpha \mathcal{U} + \max\left\{0, E_{\omega,0}^{(N)} - E_{\omega,0}^{(N-1)}\right\} \times 16 m_e^2 \tilde{\mathcal{E}}. \quad \text{(F.30)}\end{aligned}$$

From the bound (F.29), we have

$$E_{\omega,0}^{(N)} - E_{\omega,0}^{(N-1)} \le \hbar\omega_c \left(\mathcal{C}_5 N^{2/3} + \mathcal{C}_6\right) \tag{F.31}$$

by choosing n as

$$N^{2/3} + 1 \ge n + \frac{1}{2} > N^{2/3} \ge N_{\min}^{2/3}. \tag{F.32}$$

Substituting (F.31) into the above (F.30), we obtain the desired bound (F.17) in Proposition F.2. ■

G Estimates of the ground state energy $E_{\omega,0}^{(N)}$ and the ground state expectation of $U^{(N)}$

The aim of this appendix is to estimate the ground state energy $E_{\omega,0}^{(N)}$ of the Hamiltonian $H_{\omega,0}^{(N)}$ of (5.2) and the expectation value of $U^{(N)}$ with respect to the ground state $\Phi_{\omega,0}^{(N)}$. The results are summarized as follows:

Lemma G.1 *Let $\Phi_{\omega,0}^{(N)}$ be the ground state eigenvector of the Hamiltonian $H_{\omega,0}^{(N)}$ with the energy eigenvalue $E_{\omega,0}^{(N)}$. Then the following two bounds are valid:*

$$\frac{E_{\omega,0}^{(N)}}{N} \le \frac{(\nu+1)^2}{2\nu}\hbar\omega_c + \|V_\omega\| + \frac{eB(\nu+1)}{h}\left[\left\|U^{(2)}\right\|_1 + \varepsilon_1(\nu)\right] \le \tilde{\mathcal{E}}, \tag{G.1}$$

and

$$\frac{1}{N}\left\langle \Phi_{\omega,0}^{(N)}, U^{(N)} \Phi_{\omega,0}^{(N)} \right\rangle \le 2\|V_\omega\| + \frac{eB(\nu+1)}{h}\left[\left\|U^{(2)}\right\|_1 + \varepsilon_1(\nu)\right] \le \mathcal{U}. \tag{G.2}$$

Here $\tilde{\mathcal{E}}$ and $\mathcal{U}$ are positive constants which are independent of the linear dimensions L_x, L_y of the system, and $\varepsilon_1(\nu)$ is a small real number which tends to zero as $L_x, L_y \to +\infty$. The norm $\|\cdots\|_1$ is defined as

$$\|f\|_1 := \int_S |f(x,y)| dxdy \tag{G.3}$$

for a function f on S.

We begin with the following lemma:

Lemma G.2 *The following two bounds are valid:*

$$E_{\omega,0}^{(N)} \le N\frac{(\nu+1)^2}{2\nu}\hbar\omega_c + N\left\|V_\omega\right\| + \left\langle \Phi_0^{(N)}, U^{(N)}\Phi_0^{(N)}\right\rangle, \tag{G.4}$$

and

$$\left\langle \Phi_{\omega,0}^{(N)}, U^{(N)}\Phi_{\omega,0}^{(N)}\right\rangle \le 2N\left\|V_\omega\right\| + \left\langle \Phi_0^{(N)}, U^{(N)}\Phi_0^{(N)}\right\rangle, \tag{G.5}$$

where the vector $\Phi_0^{(N)}$ is the N electrons ground state eigenvector of the non-interacting Hamiltonian

$$\sum_{j=1}^{N}\mathcal{H}_j = \sum_{j=1}^{N}\frac{1}{2m_e}\left[(p_{x,j}-eBy_j)^2 + p_{y,j}^2\right] \tag{G.6}$$

with the periodic boundary conditions.

Proof: By definition, we have

$$\begin{aligned} E_{\omega,0}^{(N)} = \left\langle \Phi_{\omega,0}^{(N)}, H_{\omega,0}^{(N)}\Phi_{\omega,0}^{(N)}\right\rangle &\le \left\langle \Phi_0^{(N)}, H_{\omega,0}^{(N)}\Phi_0^{(N)}\right\rangle \\ &\le \sum_{j=1}^{N}\left\langle \Phi_0^{(N)}, \mathcal{H}_j\Phi_0^{(N)}\right\rangle + N\left\|V_\omega\right\| + \left\langle \Phi_0^{(N)}, U^{(N)}\Phi_0^{(N)}\right\rangle. \end{aligned} \tag{G.7}$$

Therefore the first bound (G.4) follows from

$$\sum_{j=1}^{N}\left\langle \Phi_0^{(N)}, \mathcal{H}_j\Phi_0^{(N)}\right\rangle = \sum_{\ell}\left(n_\ell + \frac{1}{2}\right)\hbar\omega_c \le M\frac{(\nu+1)^2}{2}\hbar\omega_c, \tag{G.8}$$

where the second sum runs over all the states ℓ in the Fermi sea. Further, by combining (G.7) with

$$\sum_{j=1}^{N}\left\langle \Phi_0^{(N)}, \mathcal{H}_j\Phi_0^{(N)}\right\rangle \le \sum_{j=1}^{N}\left\langle \Phi_{\omega,0}^{(N)}, \mathcal{H}_j\Phi_{\omega,0}^{(N)}\right\rangle, \tag{G.9}$$

we get the second bound (G.5). ■

Owing to this lemma, it is sufficient to estimate the expectation $\left\langle \Phi_0^{(N)}, U^{(N)}\Phi_0^{(N)}\right\rangle$. For this purpose, we use the following lemma:

Lemma G.3 *Let $\phi_{n,k}^{\mathrm{P}}$ be the eigenvectors (3.17) of the single electron Hamiltonian $\mathcal{H}$ of (3.1) with the peridic boundary conditions (3.9). Then*

$$\sum_k \int_S dx_i dy_i\; U^{(2)}(x_i - x_j, y_i - y_j)\left|\phi_{n,k}^{\mathrm{P}}(x_i, y_i)\right|^2 = \frac{eB}{h}\left[\left\|U^{(2)}\right\|_1 + \varepsilon_1^{(n)}\right] \tag{G.10}$$

for any $(x_j, y_j) \in \mathbf{R}^2$. Here the sum is over all the wavenumbers k for a fixed Landau index n, and the small real number $\varepsilon_1^{(n)}$ tends to zero uniformly in the Landau index n as $L_x, L_y \to +\infty$.

Proof: Consider the function

$$\rho_n(x,y) := \sum_k \left|\phi^{\mathrm{P}}_{n,k}(x,y)\right|^2. \tag{G.11}$$

From the definition (3.17) of the vector $\phi^{\mathrm{P}}_{n,k}$, the function ρ_n is periodic in both x and y directions as

$$\rho_n(x,y) = \rho_n(x+\Delta x, y) = \rho_n(x, y+\Delta y), \tag{G.12}$$

where

$$\Delta x = \frac{h}{eB}\frac{1}{L_y} \quad \text{and} \quad \Delta y = \frac{h}{eB}\frac{1}{L_x}. \tag{G.13}$$

From this periodicity and the periodicity (2.10) of the two-body interaction $U^{(2)}$, we can assume $|x_j| \le \Delta x/2, |y_j| \le \Delta y/2$. The integral of ρ_n on the unit cell $\Delta_{\ell,m}$ becomes

$$\int_{\Delta_{\ell,m}} dxdy\ \rho_n(x,y) = \frac{1}{M}, \tag{G.14}$$

where

$$\Delta_{\ell,m} := [x_\ell, x_{\ell+1}] \times [y_m, y_{m+1}] \tag{G.15}$$

with

$$x_\ell = -\frac{L_x}{2} + (\ell-1)\Delta x \quad \text{for } \ell = 1,2,\dots,M \tag{G.16}$$

and

$$y_m = -\frac{L_y}{2} + (m-1)\Delta y \quad \text{for } m = 1,2,\dots,M. \tag{G.17}$$

Since the function $U^{(2)}$ is continuous by the assumption, there exists a point $(\xi^{\ell,m}, \eta^{\ell,m}) \in \Delta_{\ell,m}$ such that

$$\begin{aligned}
&\int_{\Delta_{\ell,m}} dx_i dy_i\ U^{(2)}(x_i - x_j, y_i - y_j)\rho_n(x_i,y_i) \\
=\ & U^{(2)}(\xi^{\ell,m} - x_j, \eta^{\ell,m} - y_j)\int_{\Delta_{\ell,m}} dx_i dy_i\ \rho_n(x_i,y_i) \\
=\ & \frac{U^{(2)}(\xi^{\ell,m} - x_j, \eta^{\ell,m} - y_j)}{M}.
\end{aligned} \tag{G.18}$$

Using (G.18) and the definitions of $\Delta x, \Delta y$, we get

$$\begin{aligned}
&\sum_k \int_S dx_i dy_i\ U^{(2)}(x_i - x_j, y_i - y_j)\left|\phi^{\mathrm{P}}_{n,k}(x_i,y_i)\right|^2 \\
=\ & \int_S dx_i dy_i\ U^{(2)}(x_i - x_j, y_i - y_j)\rho_n(x_i,y_i) \\
=\ & \frac{eB}{h}\sum_{\ell,m} U^{(2)}(\xi^{\ell,m} - x_j, \eta^{\ell,m} - y_j)\Delta x \Delta y \\
=\ & \frac{eB}{h}\sum_{\sqrt{(\xi^{\ell,m})^2 + (\eta^{\ell,m})^2} \le R'} U^{(2)}(\xi^{\ell,m} - x_j, \eta^{\ell,m} - y_j)\Delta x \Delta y \\
+\ & \frac{eB}{h}\sum_{\sqrt{(\xi^{\ell,m})^2 + (\eta^{\ell,m})^2} > R'} U^{(2)}(\xi^{\ell,m} - x_j, \eta^{\ell,m} - y_j)\Delta x \Delta y
\end{aligned} \tag{G.19}$$

with a large positive number R'. Since $U^{(2)}$ is continuous, the first term in the last line converges to

$$\frac{eB}{h}\int_{\sqrt{(x_i)^2+(y_i)^2}\le R'} dx_i dy_i\, U^{(2)}(x_i-x_j, y_i-y_j) \tag{G.20}$$

as $L_x, L_y \to +\infty$. The second term is vanishing uniformly in n, L_x, L_y as $R' \to +\infty$ from the assumption (2.12) about $U^{(2)}$. Thus the statement of the lemma is proved. ■

Proof of Lemma G.1: Note that

$$\begin{aligned}
&\left\langle \Phi_0^{(N)}, U^{(N)}\Phi_0^{(N)}\right\rangle \\
= &\ \frac{1}{2}\sum_{m,k,n,k'}\int dx_i dy_i \int dx_j dy_j\ \left[\phi_{m,k}^{\mathrm{P}}(\mathbf{r}_i)\right]^* \left[\phi_{n,k'}^{\mathrm{P}}(\mathbf{r}_j)\right]^* U^{(2)}(\mathbf{r}_{ij})\phi_{m,k}^{\mathrm{P}}(\mathbf{r}_i)\phi_{n,k'}^{\mathrm{P}}(\mathbf{r}_j) \\
- &\ \frac{1}{2}\sum_{m,k,n,k'}\int dx_i dy_i \int dx_j dy_j\ \left[\phi_{m,k}^{\mathrm{P}}(\mathbf{r}_i)\right]^* \left[\phi_{n,k'}^{\mathrm{P}}(\mathbf{r}_j)\right]^* U^{(2)}(\mathbf{r}_{ij})\phi_{n,k'}^{\mathrm{P}}(\mathbf{r}_i)\phi_{m,k}^{\mathrm{P}}(\mathbf{r}_j) \\
\le &\ \sum_{m,k,n,k'}\int dx_i dy_i \int dx_j dy_j\ U^{(2)}(\mathbf{r}_{ij})\left|\phi_{m,k}^{\mathrm{P}}(\mathbf{r}_i)\right|^2 \left|\phi_{n,k'}^{\mathrm{P}}(\mathbf{r}_j)\right|^2,
\end{aligned} \tag{G.21}$$

where we have written $\mathbf{r}_{ij} = (x_i - x_j, y_i - y_j)$ for simplicity.

On the other hand we have

$$\sum_{m,k}\int dx_i dy_i\ U^{(2)}(\mathbf{r}_{ij})\left|\phi_{m,k}^{\mathrm{P}}(\mathbf{r}_i)\right|^2 \le \frac{eB(\nu+1)}{h}\left[\left\|U^{(2)}\right\|_1 + \varepsilon_1(\nu)\right] \tag{G.22}$$

from Lemma G.3. Here $\varepsilon_1(\nu)$ is a small real number which tends to zero as $L_x, L_y \to +\infty$. Substituting this inequality into the right-hand side of (G.21), we gat

$$\left\langle \Phi_0^{(N)}, U^{(N)}\Phi_0^{(N)}\right\rangle \le N\left\{\frac{eB(\nu+1)}{h}\left[\left\|U^{(2)}\right\|_1 + \varepsilon_1(\nu)\right]\right\}. \tag{G.23}$$

Combining this with Lemma G.2, we obtain the bounds in Lemma G.1. ■

H Estimate of $E_{\omega,0}^{(N)} - E_{\omega,0}^{(N-1)}$

In this appendix, we prove Lemma F.3. For this purpose we consider

$$E_{\omega,0}^{(N)} \le \frac{\eta\left(H_{\omega,0}^{(N)}\right)}{\eta(1)}, \tag{H.1}$$

where

$$\eta(\cdots) = \frac{1}{M}\sum_k \left\langle \Phi_{\omega,(n,k)}^{(N)}, (\cdots)\Phi_{\omega,(n,k)}^{(N)}\right\rangle \tag{H.2}$$

with

$$\Phi_{\omega,(n,k)}^{(N)} = \mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)} \otimes \phi_{n,k}^{\mathrm{P}}\right]. \tag{H.3}$$

Here Asym$[\cdots]$ is the antisymmetrization of a wavefunction, whose definition is given in (C.3), $\Phi_{\omega,0}^{(N-1)}$ is the $N-1$ electrons ground state eigenvector of the Hamiltonian $H_{\omega,0}^{(N-1)}$ with norm one, and $\phi_{n,k}^{\mathrm{P}}$ are the normalized eigenvectors (3.17) of the single electron Hamiltonian $\mathcal{H}$ of (3.1) with the periodic boundary conditions (3.9). We introduce an orthogonal decomposition of the vector $\Phi_{\omega,0}^{(N-1)}$ as

$$\Phi_{\omega,0}^{(N-1)} = \Psi_{1,(n,k)}^{(N-1)} + \Psi_{2,(n,k)}^{(N-1)} \tag{H.4}$$

with

$$\Psi_{1,(n,k)}^{(N-1)} = \prod_{j=1}^{N-1} \left[1 - P_j^{(n,k)}\right] \Phi_{\omega,0}^{(N-1)}, \tag{H.5}$$

where $P^{(n,k)}$ is the orthogonal projection onto the vector $\phi_{n,k}^{\mathrm{P}}$. Then $\mathrm{Asym}\left[\Psi_{2,(n,k)}^{(N-1)} \otimes \phi_{n,k}^{\mathrm{P}}\right]$ is identically zero because the vector $\Phi_{\omega,0}^{(N-1)}$ is expanded as

$$\Phi_{\omega,0}^{(N-1)} = \sum_{\{\xi_j\}} a_{\{\xi_j\}} \mathrm{Asym}\left[\phi_{\xi_1}^{\mathrm{P}} \otimes \phi_{\xi_2}^{\mathrm{P}} \otimes \cdots \otimes \phi_{\xi_{N-1}}^{\mathrm{P}}\right] \tag{H.6}$$

in terms of the vectors $\{\phi_{n,k}^{\mathrm{P}}\}$. Here we denote by ξ the pair of a Landau index n and a wavenumber k, i.e., $\xi_j = (n_j, k_j)$. From this observation, we have

$$\begin{aligned}
\eta(1) &= \frac{1}{M} \sum_k \left\langle \Phi_{\omega,(n,k)}^{(N)}, \Phi_{\omega,(n,k)}^{(N)} \right\rangle \\
&= \frac{1}{M} \sum_k \left\langle \mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)} \otimes \phi_{n,k}^{\mathrm{P}}\right], \mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)} \otimes \phi_{n,k}^{\mathrm{P}}\right] \right\rangle \\
&= \frac{1}{M} \sum_k \left\langle \mathrm{Asym}\left[\Psi_{1,(n,k)}^{(N-1)} \otimes \phi_{n,k}^{\mathrm{P}}\right], \mathrm{Asym}\left[\Psi_{1,(n,k)}^{(N-1)} \otimes \phi_{n,k}^{\mathrm{P}}\right] \right\rangle, \\
&= \frac{1}{M} \sum_k \left\| \Psi_{1,(n,k)}^{(N-1)} \right\|^2 = 1 - \frac{1}{M} \sum_k \left\| \Psi_{2,(n,k)}^{(N-1)} \right\|^2 .
\end{aligned} \tag{H.7}$$

Lemma H.1 *The following bound is valid:*

$$\frac{1}{M} \sum_k \left\| \Psi_{2,(n,k)}^{(N-1)} \right\|^2 \le \frac{N_{\min}^{2/3}}{2n+1} \tag{H.8}$$

with the positive constant $N_{\min}$ is given by (F.16).

Proof: By definition, we have

$$\left\| \Psi_{2,(n,k)}^{(N-1)} \right\|^2 = \sum_{\{\xi_j\}} \left| a_{\{\xi_j\}} \right|^2 \sum_{\xi' \in \{\xi_1, \xi_2, \ldots, \xi_{N-1}\}} \left(\phi_{\xi'}, P^{(n,k)} \phi_{\xi'} \right). \tag{H.9}$$

Clearly,

$$\frac{1}{M} \sum_k \left\| \Psi_{2,(n,k)}^{(N-1)} \right\|^2 = \frac{1}{M} \sum_{\{\xi_j\}} \left| a_{\{\xi_j\}} \right|^2 \sum_{\xi' \in \{\xi_1, \xi_2, \ldots, \xi_{N-1}\}} \left(\phi_{\xi'}, P^{(n)} \phi_{\xi'} \right), \tag{H.10}$$

where $P^{(n)} = \sum_k P^{(n,k)}$, i.e., the orthogonal projection onto the Landau level with the index n.

On the other hand we have, for the ground state energy $E_{\omega,0}^{(N-1)}$,

$$\begin{aligned}
\frac{E_{\omega,0}^{(N-1)}}{N-1} &= \frac{1}{N-1}\left\langle \Phi_{\omega,0}^{(N-1)}, H_{\omega,0}^{(N-1)}\Phi_{\omega,0}^{(N-1)}\right\rangle \\
&= \frac{1}{N-1}\sum_{j=1}^{N-1}\left\langle \Phi_{\omega,0}^{(N-1)}, \mathcal{H}_j\Phi_{\omega,0}^{(N-1)}\right\rangle + \frac{1}{N-1}\sum_{j=1}^{N-1}\left\langle \Phi_{\omega,0}^{(N-1)}, V_\omega(\mathbf{r}_j)\Phi_{\omega,0}^{(N-1)}\right\rangle \\
&+ \frac{1}{N-1}\left\langle \Phi_{\omega,0}^{(N-1)}, U^{(N-1)}\Phi_{\omega,0}^{(N-1)}\right\rangle \\
&\ge \frac{1}{N-1}\sum_{j=1}^{N-1}\left\langle \Phi_{\omega,0}^{(N-1)}, P_j^{(n)}\mathcal{H}_j\Phi_{\omega,0}^{(N-1)}\right\rangle - \|V_\omega\| \\
&= \frac{1}{N-1}\hbar\omega_c\left(n+\frac{1}{2}\right)\sum_{j=1}^{N-1}\left\langle \Phi_{\omega,0}^{(N-1)}, P_j^{(n)}\Phi_{\omega,0}^{(N-1)}\right\rangle - \|V_\omega\| \\
&= \frac{1}{N-1}\hbar\omega_c\left(n+\frac{1}{2}\right)\sum_{\{\xi_j\}}\left|a_{\{\xi_j\}}\right|^2 \sum_{\xi'\in\{\xi_1,\xi_2,\ldots,\xi_{N-1}\}}\left(\phi_{\xi'}, P^{(n)}\phi_{\xi'}\right) - \|V_\omega\|\,.
\end{aligned} \tag{H.11}$$

Combining this with the above (H.10), we get

$$\frac{1}{M}\sum_k\left\|\Psi_{2,(n,k)}^{(N-1)}\right\|^2 \le \frac{\nu}{\hbar\omega_c}\left(\|V_\omega\| + \frac{E_{\omega,0}^{(N-1)}}{N-1}\right)\frac{1}{n+1/2}. \tag{H.12}$$

Using the bound (G.1) in Lemma G.1 in the preceding appendix and $N_{\min}$ of (F.16), we obtain the desired bound (H.8). ■

From (H.7) and (H.8), we have

Corollary H.2

$$\eta(1) \ge \frac{1}{2} \quad \textit{for} \quad n + \frac{1}{2} \ge N_{\min}^{2/3}. \tag{H.13}$$

Next consider the numerator of the right-hand side of (H.1),

$$\eta\left(H_{\omega,0}^{(N)}\right) = \frac{1}{M}\sum_k\left\langle \Phi_{\omega,(n,k)}^{(N)}, H_{\omega,0}^{(N)}\Phi_{\omega,(n,k)}^{(N)}\right\rangle. \tag{H.14}$$

By definition, we have

$$\begin{aligned}
\left\langle \Phi_{\omega,(n,k)}^{(N)}, H_{\omega,0}^{(N)}\Phi_{\omega,(n,k)}^{(N)}\right\rangle &= \left\langle \mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right], H_{\omega,0}^{(N)}\mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle \\
&= \left\langle \mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right], \mathrm{Asym}\left[H_{\omega,0}^{(N)}\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle \\
&= E_{\omega,0}^{(N-1)}\left\|\Phi_{\omega,(n,k)}^{(N)}\right\|^2 + \hbar\omega_c\left(n+\frac{1}{2}\right)\left\|\Phi_{\omega,(n,k)}^{(N)}\right\|^2 \\
&+ \left\langle \mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right], \mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes V_\omega\phi_{n,k}^{\mathrm{P}}\right]\right\rangle \\
&+ \left\langle \mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right], \mathrm{Asym}\left[U^{(N,\cdots)}\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle,
\end{aligned} \tag{H.15}$$

where the operator $U^{(N,\cdots)}$ is defined as

$$\begin{aligned}&\left(U^{(N,j)}\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right)(\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_{j-1},\mathbf{r}_{j+1},\dots,\mathbf{r}_N,\mathbf{r}_j)\\ =\ &\sum_{i\neq j}U^{(2)}(x_i-x_j,y_i-y_j)\Phi_{\omega,0}^{(N-1)}(\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_{j-1},\mathbf{r}_{j+1},\dots,\mathbf{r}_N)\phi_{n,k}^{\mathrm{P}}(\mathbf{r}_j).\end{aligned}\tag{H.16}$$

Substituting this into the right-hand side of (H.14), we obtain

$$\begin{aligned}\eta\left(H_{\omega,0}^{(N)}\right)\ =\ &\left[E_{\omega,0}^{(N-1)}+\hbar\omega_c\left(n+\frac{1}{2}\right)\right]\eta(1)\\ +\ &\frac{1}{M}\sum_k\left\langle\mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right],\mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes V_\omega\phi_{n,k}^{\mathrm{P}}\right]\right\rangle\\ +\ &\frac{1}{M}\sum_k\left\langle\mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right],\mathrm{Asym}\left[U^{(N,\cdots)}\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle.\end{aligned}\tag{H.17}$$

The first sum in the right-hand side is written as

$$\begin{aligned}&\frac{1}{M}\sum_k\left\langle\mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right],\mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes V_\omega\phi_{n,k}^{\mathrm{P}}\right]\right\rangle\\ =\ &\frac{1}{M}\sum_k\left\langle\mathrm{Asym}\left[\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right],\mathrm{Asym}\left[\Psi_{3,(n,k)}^{(N-1)}\otimes V_\omega\phi_{n,k}^{\mathrm{P}}\right]\right\rangle,\end{aligned}\tag{H.18}$$

where

$$\Psi_{3,(n,k)}^{(N-1)}=\prod_{j=1}^{N-1}\left[1-P_j^{(n,k)}(V_\omega)\right]\Phi_{\omega,0}^{(N-1)}.\tag{H.19}$$

Here $P^{(n,k)}(V_\omega)$ is the orthogonal projection onto the vector $V_\omega\phi_{n,k}^{\mathrm{P}}$. Using the Schwarz inequality, we have

$$\begin{aligned}&\left|\frac{1}{M}\sum_k\left\langle\mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right],\mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes V_\omega\phi_{n,k}^{\mathrm{P}}\right]\right\rangle\right|\\ \le\ &\frac{1}{M}\sum_k\left\|\Psi_{1,(n,k)}^{(N-1)}\right\|\ \left\|\Psi_{3,(n,k)}^{(N-1)}\right\|\ \left\|V_\omega\phi_{n,k}^{\mathrm{P}}\right\|\\ \le\ &\frac{1}{M}\sum_k\left\|\Phi_{\omega,0}^{(N-1)}\right\|^2\|V_\omega\|=\|V_\omega\|.\end{aligned}\tag{H.20}$$

The second sum in the right-hand side of (H.17) is written as

$$\begin{aligned}&\frac{1}{M}\sum_k\left\langle\mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right],\mathrm{Asym}\left[U^{(N,\cdots)}\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle\\ =\ &\frac{1}{M}\sum_k\left\langle\mathrm{Asym}\left[\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right],\mathrm{Asym}\left[U^{(N,\cdots)}\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle\\ +\ &\frac{1}{M}\sum_k\left\langle\mathrm{Asym}\left[\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right],\mathrm{Asym}\left[U^{(N,\cdots)}\Psi_{2,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle\end{aligned}\tag{H.21}$$

by using the decomposition (H.4). This second sum in the right-hand side is evaluated as follows:

Lemma H.3 *The following bound is valid:*

$$\begin{aligned}
&\left|\frac{1}{M}\sum_{k}\left\langle \mathrm{Asym}\left[\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right], \mathrm{Asym}\left[U^{(N,\cdots)}\Psi_{2,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle\right| \\
\le\ & N_{\min}^{1/3}\sqrt{(\varepsilon_2+\alpha/\pi)\left\|U^{(2)}\right\|_1\left\|U^{(2)}\right\|_\infty}\times\frac{N-1}{\sqrt{2n+1}}, \qquad \text{(H.22)}
\end{aligned}$$

where α is the positive constant given in the assumption (2.11) on the interaction $U^{(2)}$, and ε_2 is a positive number which tends to zero as $L_x, L_y \to +\infty$. The norm $\|\cdots\|_\infty$ is given by

$$\|f\|_\infty := \sup_{(x,y)\in S}|f(x,y)| \qquad \text{(H.23)}$$

for a continuous function f on S.

Proof: The interaction potential $U^{(2)}$ is written as

$$U^{(2)}(x_j-x_\ell, y_j-y_\ell)=\frac{1}{\sqrt{L_xL_y}}\sum_{k_x,k_y}\hat{U}^{(2)}(k_x,k_y)e^{ik_xx_j+ik_yy_j}e^{-ik_xx_\ell-ik_yy_\ell} \qquad \text{(H.24)}$$

in terms of the Fourier transform of $\hat{U}^{(2)}$. Clearly,

$$\begin{aligned}
&U^{(N,j)}(\mathbf{r}_j;\mathbf{r}_1,\ldots,\mathbf{r}_{j-1},\mathbf{r}_{j+1},\ldots,\mathbf{r}_N) \\
=\ &\sum_{\ell\ne j}U^{(2)}(x_j-x_\ell,y_j-y_\ell) \\
=\ &\frac{1}{\sqrt{L_xL_y}}\sum_{k_x,k_y}\hat{U}^{(2)}(k_x,k_y)e^{ik_xx_j+ik_yy_j}\sum_{\ell\ne j}e^{-ik_xx_\ell-ik_yy_\ell}. \qquad \text{(H.25)}
\end{aligned}$$

Using this expression, we have

$$\begin{aligned}
&\frac{1}{M}\sum_{k}\left\langle \mathrm{Asym}\left[\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right], \mathrm{Asym}\left[U^{(N,\cdots)}\Psi_{2,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle \\
&=\frac{1}{M}\sum_{k_x,k_y,k}\frac{\hat{U}^{(2)}(k_x,k_y)}{\sqrt{L_xL_y}}\left\langle \mathrm{Asym}\left[\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right], \mathrm{Asym}\left[\tilde{\Psi}_{2,(n,k)}^{(N-1)}(k_x,k_y)\otimes\tilde{\phi}_{n,k}^{\mathrm{P}}(k_x,k_y)\right]\right\rangle, \qquad \text{(H.26)}
\end{aligned}$$

where

$$\tilde{\Psi}_{2,(n,k)}^{(N-1)}(k_x,k_y)=\left\{\prod_{\ell\ne j}\left[1-\tilde{P}_\ell^{(n,k)}(k_x,k_y)\right]\right\}\sum_{\ell\ne j}e^{-ik_xx_\ell-ik_yy_\ell}\Psi_{2,(n,k)}^{(N-1)}, \qquad \text{(H.27)}$$

and

$$\tilde{\phi}_{n,k}^{\mathrm{P}}(k_x,k_y)=e^{ik_xx+ik_yy}\phi_{n,k}^{\mathrm{P}}. \qquad \text{(H.28)}$$

Here $\tilde{P}^{(n,k)}(k_x, k_y)$ is the projection onto the vector $\tilde{\phi}^{\rm P}_{n,k}(k_x, k_y)$. Applying the Schwarz inequality to the right-hand side of (H.26), we have

$$\begin{aligned}
&\left|\frac{1}{M}\sum_k \left\langle \mathrm{Asym}\left[\Psi^{(N-1)}_{1,(n,k)} \otimes \phi^{\rm P}_{n,k}\right], \mathrm{Asym}\left[U^{(N,\cdots)}\Psi^{(N-1)}_{2,(n,k)} \otimes \phi^{\rm P}_{n,k}\right]\right\rangle\right|^2 \\
\le\ & \left[\sum_{k_x,k_y}(k_x^2+k_y^2+\alpha)^2\left|\hat{U}^{(2)}(k_x,k_y)\right|^2 \frac{1}{M}\sum_k \left\|\Psi^{(N-1)}_{1,(n,k)}\right\|^2\right] \\
\times\ & \left[\frac{1}{L_xL_y}\sum_{k_x,k_y}\frac{1}{(k_x^2+k_y^2+\alpha)^2}\frac{1}{M}\sum_k \left\|\tilde{\Psi}^{(N-1)}_{2,(n,k)}(k_x,k_y)\right\|^2\right] \\
\le\ & 4\alpha^2\left\|U^{(2)}\right\|_1\left\|U^{(2)}\right\|_\infty \times \left(\frac{1}{4\pi\alpha}+\varepsilon_2'\right)\times (N-1)^2\frac{1}{M}\sum_k\left\|\Psi^{(N-1)}_{2,(n,k)}\right\|^2, \qquad \text{(H.29)}
\end{aligned}$$

where we have used the following three bounds:

$$\left\|\tilde{\Psi}^{(N-1)}_{2,(n,k)}(k_x,k_y)\right\|^2 \le (N-1)^2\left\|\Psi^{(N-1)}_{2,(n,k)}\right\|^2, \qquad \text{(H.30)}$$

$$\begin{aligned}
\sum_{k_x,k_y}(k_x^2+k_y^2+\alpha)^2\left|\hat{U}^{(2)}(k_x,k_y)\right|^2 &= \int dx_j dy_j \left|\left(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2}+\alpha\right)U^{(2)}(\mathbf{r}_{j\ell})\right|^2 \\
&\le 4\alpha^2\int dx_j dy_j\left|U^{(2)}(\mathbf{r}_{j\ell})\right|^2 \\
&\le 4\alpha^2\left\|U^{(2)}\right\|_1\left\|U^{(2)}\right\|_\infty, \qquad \text{(H.31)}
\end{aligned}$$

and

$$\frac{1}{L_xL_y}\sum_{k_x,k_y}\frac{1}{(k_x^2+k_y^2+\alpha)^2} = \frac{1}{4\pi\alpha}+\varepsilon_2'. \qquad \text{(H.32)}$$

Clearly ε_2' defined by the above equation is a real number which tends to zero as $L_x, L_y \to +\infty$. The bound (H.31) is easily derived from the assumption (2.11) about $U^{(2)}$. Combining (H.29) with (H.8), we get the desired bound (H.22). ■

The first sum in the right-hand side of (H.21) is evaluated as follows:

Lemma H.4 *The following bound is valid:*

$$\begin{aligned}
&\left|\frac{1}{M}\sum_k \left\langle \mathrm{Asym}\left[\Psi^{(N-1)}_{1,(n,k)} \otimes \phi^{\rm P}_{n,k}\right], \mathrm{Asym}\left[U^{(N,\cdots)}\Psi^{(N-1)}_{1,(n,k)} \otimes \phi^{\rm P}_{n,k}\right]\right\rangle\right| \\
\le\ & \frac{4\nu eB}{h}\left[\left\|U^{(2)}\right\|_1+\varepsilon_1^{(n)}\right]+4N_{\min}^{2/3}\left\|U^{(2)}\right\|_\infty\frac{N-1}{2n+1}, \qquad \text{(H.33)}
\end{aligned}$$

where $\varepsilon_1^{(n)}$ is a positive number which tends to zero as $L_x, L_y \to \infty$.

Proof: Note that

$$\begin{aligned}
&\frac{1}{M}\sum_k \left\langle \mathrm{Asym}\left[\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right], \mathrm{Asym}\left[U^{(N,\cdots)}\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle \\
= \;& \frac{N-1}{M}\sum_k \int dv^{(N)} \left|\Psi_{1,(n,k)}^{(N-1)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N-1})\right|^2 U^{(2)}(x_{N-1}-x_N, y_{N-1}-y_N)\left|\phi_{n,k}^{\mathrm{P}}(\mathbf{r}_N)\right|^2 \\
- \;& \frac{N-1}{M}\sum_k \int dv^{(N)} \left[\Psi_{1,(n,k)}^{(N-1)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N-2},\mathbf{r}_N)\right]^* \left[\phi_{n,k}^{\mathrm{P}}(\mathbf{r}_{N-1})\right]^* \\
& \qquad \times U^{(2)}(x_{N-1}-x_N, y_{N-1}-y_N)\Psi_{1,(n,k)}^{(N-1)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N-2},\mathbf{r}_{N-1})\phi_{n,k}^{\mathrm{P}}(\mathbf{r}_N). \qquad \text{(H.34)}
\end{aligned}$$

Here $dv^{(N)} = dx_1dy_1dx_2dy_2\cdots dx_Ndy_N$. Since the absolute value of the second term in the right-hand side of (H.34) is bounded by the first term by using the Schwarz inequality, we have an inequality

$$\begin{aligned}
&\left|\frac{1}{M}\sum_k \left\langle \mathrm{Asym}\left[\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right], \mathrm{Asym}\left[U^{(N,\cdots)}\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle\right| \\
\le \;& \frac{2(N-1)}{M}\sum_k \int dv^{(N)} \left|\Psi_{1,(n,k)}^{(N-1)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N-1})\right|^2 U^{(2)}(x_{N-1}-x_N, y_{N-1}-y_N)\left|\phi_{n,k}^{\mathrm{P}}(\mathbf{r}_N)\right|^2. \qquad \text{(H.35)}
\end{aligned}$$

Note that

$$\left|\Psi_{1,(n,k)}^{(N-1)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N-1})\right|^2 \le 2\left[\left|\Phi_{\omega,0}^{(N-1)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N-1})\right|^2 + \left|\Psi_{2,(n,k)}^{(N-1)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N-1})\right|^2\right] \qquad \text{(H.36)}$$

which is easily obtained by using the decomposition $\Phi_{\omega,0}^{(N-1)} = \Psi_{1,(n,k)}^{(N-1)} + \Psi_{2,(n,k)}^{(N-1)}$. Substituting this inequality into the right-hand side of (H.35), we get

$$\begin{aligned}
&\left|\frac{1}{M}\sum_k \left\langle \mathrm{Asym}\left[\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right], \mathrm{Asym}\left[U^{(N,\cdots)}\Psi_{1,(n,k)}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle\right| \\
\le \;& \frac{4(N-1)}{M}\sum_k \int dv^{(N)} \left|\Phi_{\omega,0}^{(N-1)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N-1})\right|^2 U^{(2)}(x_{N-1}-x_N, y_{N-1}-y_N)\left|\phi_{n,k}^{\mathrm{P}}(\mathbf{r}_N)\right|^2 \\
+ \;& \frac{4(N-1)}{M}\sum_k \int dv^{(N)} \left|\Psi_{2,(n,k)}^{(N-1)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N-1})\right|^2 U^{(2)}(x_{N-1}-x_N, y_{N-1}-y_N)\left|\phi_{n,k}^{\mathrm{P}}(\mathbf{r}_N)\right|^2 \\
\le \;& \frac{4(N-1)}{M}\sum_k \int dv^{(N)} \left|\Phi_{\omega,0}^{(N-1)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N-1})\right|^2 U^{(2)}(x_{N-1}-x_N, y_{N-1}-y_N)\left|\phi_{n,k}^{\mathrm{P}}(\mathbf{r}_N)\right|^2 \\
+ \;& 4(N-1)\left\|U^{(2)}\right\|_\infty \frac{1}{M}\sum_k \left\|\Psi_{2,(n,k)}^{(N-1)}\right\|^2. \qquad \text{(H.37)}
\end{aligned}$$

Combining this with (H.8) and Lemma G.3, we obtain the desired result (H.33). ■

Proof of Lemma F.3: Combining (H.21), (H.22), (H.33), we have

$$\left|\frac{1}{M}\sum_k \left\langle \mathrm{Asym}\left[\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right], \mathrm{Asym}\left[U^{(N,\cdots)}\Phi_{\omega,0}^{(N-1)}\otimes\phi_{n,k}^{\mathrm{P}}\right]\right\rangle\right|$$

$$\leq \quad N_{\min}^{1/3}\left[\sqrt{(\varepsilon_2+\alpha/\pi)\,\|U^{(2)}\|_1\,\|U^{(2)}\|_\infty}+\frac{4N_{\min}^{1/3}\left\|U^{(2)}\right\|_\infty}{\sqrt{2n+1}}\right]\frac{N-1}{\sqrt{2n+1}}$$
$$+ \quad \frac{4\nu eB}{h}\left[\left\|U^{(2)}\right\|_1+\varepsilon_1^{(n)}\right]. \tag{H.38}$$

Combining this, (H.17) and (H.20), we obtain

$$\eta\left(H_{\omega,0}^{(N)}\right) \leq \left[E_{\omega,0}^{(N-1)}+\hbar\omega_c\left(n+\frac{1}{2}\right)\right]\eta(1)+\|V_\omega\|$$
$$+ \quad \frac{4\nu eB}{h}\left[\left\|U^{(2)}\right\|_1+\varepsilon_1^{(n)}\right]+\mathcal{C}_5'\frac{N-1}{\sqrt{2n+1}}, \tag{H.39}$$

where $\mathcal{C}_5'$ is a positive constant. Substituting this into the right-hand of (H.1), we get

$$E_{\omega,0}^{(N)}-E_{\omega,0}^{(N-1)}$$
$$\leq \quad \hbar\omega_c\left(n+\frac{1}{2}\right)+2\|V_\omega\|+\frac{8\nu eB}{h}\left[\left\|U^{(2)}\right\|_1+\varepsilon_1^{(n)}\right]+2\mathcal{C}_5'\frac{N-1}{\sqrt{2n+1}} \tag{H.40}$$

for $n+1/2 \geq N_{\min}^{2/3}$, where we have used (H.13). ■

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